Soviet-era science, translated into English

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1963

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Abstract

Full Text

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HARMONIC FUNCTIONS ON $GL(2)$

(Presented by Academician I. G. Petrovskii, 24 VI 1963)

We consider the full complex linear group of second order $G = GL(2)$ with elements

$$g = \begin{Vmatrix} g_1 & g_2 \\ g_3 & g_4 \end{Vmatrix}.$$

By the **Laplace operator** on it we shall mean the second-order differential operator, invariant under the operators of right and left translations,

$$\mathcal{L} \det \bar{g} \left(\frac{\partial^2}{\partial g_1\, \partial g_4} - \frac{\partial^2}{\partial g_2\, \partial g_3} \right),$$

where

$$\frac{\partial}{\partial g_k} = \frac{\partial}{\partial u_k} + i \frac{\partial}{\partial v_k}, \qquad g_k = u_k + i v_k.$$

More precisely, on G there are two Laplace operators, namely:

$$\mathcal{L}_1 = \operatorname{Re} \mathcal{L}, \qquad \mathcal{L}_2 = \operatorname{Im} \mathcal{L}.$$

A twice differentiable function $f(g)$ on G satisfying the two Laplace equations

$$\mathcal{L}_1 f(g) = 0, \qquad \mathcal{L}_2 f(g) = 0,$$

is called **harmonic**.

Let G_0 be the full complex unimodular linear group of second order with elements

$$g_0 = \begin{Vmatrix} g_1^0 & g_2^0 \\ g_3^0 & g_4^0 \end{Vmatrix}, \qquad \det g_0 = 1,$$

and let A be the subgroup of G_0 consisting of matrices of the form

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$$\hat{a} = \begin{Vmatrix} 1 & a \\ 0 & 1 \end{Vmatrix}.$$

The homogeneous space $H = G_0/A$ can be interpreted as the space of linear elements of the complex plane and identified with the space of triangular unimodular matrices

$$h = \begin{Vmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{Vmatrix} \begin{Vmatrix} 1 & 0 \\ z & 1 \end{Vmatrix} = \begin{Vmatrix} \xi^{-1} & 0 \\ \xi z & \xi \end{Vmatrix}^{*},$$

in which the group G_0 acts as follows:

$$h = \begin{Vmatrix} \xi^{-1} & 0 \\ \xi z & \xi \end{Vmatrix} \xrightarrow{g_0} h^{*} = h \circ g_0 = \begin{Vmatrix} \xi^{-1}(zg_2^0 + g_4^0)^{-1} & 0 \\ \xi(zg_1^0 + g_3^0) & \xi(zg_2^0 + g_4^0) \end{Vmatrix}.$$

We shall refer the matrices h to the parameters z, ξ:

$$h = (z, \xi), \qquad h \circ g_0 = \left(\frac{zg_1^0 + g_3^0}{zg_2^0 + g_4^0}, \xi(zg_2^0 + g_4^0) \right).$$

We extend the group of transformations G_0 of the space H to the group G, putting, for $g \in G$,

$$h \circ g = \begin{Vmatrix} \xi^{-1}(zg_2 + g_4)^{-1} & 0 \\ \xi(zg_1 + g_3) & \xi(zg_2 + g_4) \end{Vmatrix} \quad \text{or} \quad h \circ g = \left(\frac{zg_1 + g_3}{zg_2 + g_4}, \xi(zg_2 + g_4) \right).$$

In what follows we consider functions $\varphi(h_1, h_2) = \varphi(z_1, z_2; \xi_1, \xi_2)$ of two linear elements $h_1 = (z_1, \xi_1)$, $h_2 = (z_2, \xi_2)$, satisfying

* For more details on this, see ([1,2]).

condition

$$\varphi(\hat{\delta} h_1, \hat{\delta} h_2) = |\delta|^{-4} \varphi(h_1, h_2), \qquad \text{where} \quad \hat{\delta} = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix},$$

or, more explicitly,

$$\varphi(z_1, z_2; \delta\xi_1, \delta\xi_2) = |\delta|^{-4} \varphi(z_1, z_2; \xi_1, \xi_2). \tag{1}$$

Let $\widetilde{\varphi}(h_1, h_2)$ be such that $\widetilde{\varphi}(z_1, z_2; 1, \xi)$ is infinitely differentiable in all its (six) real arguments and

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$$\widetilde{\varphi}(z_1, z_2; 1, \xi) \equiv 0$$

whenever at least one of the inequalities

$$|z_1| > N, \qquad |z_2| > N, \qquad |\xi| > N, \qquad |\xi| < \varepsilon,$$

is satisfied, where N, ε are some positive numbers.

A function $\varphi(h_1, h_2)$ representable as a sum with a finite number of terms

$$\varphi(h_1, h_2) = \varphi_1(h_1 \circ g_1', h_2 \circ g_2') + \varphi_2(h_1 \circ g_1'', h_2 \circ g_2'') + \cdots,$$

where $\varphi_1(h_1, h_2), \varphi_2(h_1, h_2), \ldots$ are functions of precisely the type just described, will be called **finite**.

Starting from a finite $\varphi(h_1, h_2)$, construct a function $f(g)$ on G by the formula

$$f(g) = |\xi|^4 \int \varphi\left(z, \frac{zg_1 + g_3}{zg_2 + g_4}; \xi, \xi(zg_2 + g_4)\right) dz, \tag{2}$$

where $dz = dx\,dy,\ z = x + iy$, and the integral is taken over the whole plane of the complex variable z (in view of (1) it does not depend on ξ).

Introducing the notation $dh = |\xi|^4 dz$, formula (2) can be rewritten in the form

$$f(g) = \int \varphi(h, h \circ g)\, dh. \tag{3}$$

The integral (3) is a relative invariant, namely,

$$\int \varphi(h \circ \tilde{g}, h \circ \tilde{g}g)\, dh = |\det \tilde{g}|^{-2} \int \varphi(h, h \circ g)\, dh.$$

The function $f(g)$, obtained by means of (2), (3), is harmonic.

Define the norm of $\varphi(h_1, h_2)$ by the formula

$$\|\varphi\|^2 = \int |\delta|^2 \left|\varphi(h_1, \hat{\delta}h_2)\right|^2 dh_1\, dh_2\, d\delta \qquad (d\delta = d\delta_1\, d\delta_2,\ \delta = \delta_1 + i\delta_2). \tag{4}$$

The integral (4) does not depend on ξ_1, ξ_2 and is a relative invariant, namely:

$$\int |\delta|^2 \left|\varphi(h_1 \circ g_1, \hat{\delta}h_2 \circ g_2)\right|^2 dh_1\, dh_2\, d\delta =$$

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$$= |\det g_1|^{-2} |\det g_2|^{-2} \int |\delta|^2 \left|\varphi(h_1, \hat{\delta} h_2)\right|^2 \, dh_1 \, dh_2 \, d\delta.$$

Denote by Φ the Hilbert space of functions $\varphi(h_1, h_2)$ with finite norm (4).

The set $\widetilde{\Phi}$ of finite $\varphi(h_1, h_2)$ is contained in Φ and is everywhere dense in it with respect to the norm (4).

Let Γ_0 be the linear space of infinitely differentiable harmonic functions $f(g)$ on G (it is not assumed that they are obtained from φ) satisfying the condition:

$$\left| \frac{\partial^{k_1+k_2}}{\partial \gamma^{k_1} \partial \overline{\gamma}^{k_2}} \frac{\partial^{p_1+p_2+q_1+q_2}}{\partial \alpha^{p_1} \partial \overline{\alpha}^{p_2} \partial \beta^{q_1} \partial \overline{\beta}^{q_2}} f(\hat{a} \sigma \hat{b}) \right| < \frac{C(a, b)}{(1 + |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\alpha\beta|^2)^{\frac{k_1+k_2+1}{2} + \varepsilon}}$$

for any fixed $\hat{a}, \hat{b}$ and $k_1 + k_2 + p_1 + p_2 + q_1 + q_2 \leqslant 2$. Here

$$\sigma = \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}, \qquad \hat{a} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad \hat{b} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \operatorname{Re} \alpha} + i \frac{\partial}{\partial \operatorname{Im} \alpha}, \qquad \frac{\partial}{\partial \beta} = \frac{\partial}{\partial \operatorname{Re} \beta} + i \frac{\partial}{\partial \operatorname{Im} \beta}, \qquad \frac{\partial}{\partial \gamma} = \frac{\partial}{\partial \operatorname{Re} \gamma} + i \frac{\partial}{\partial \operatorname{Im} \gamma}.$$

I. For $f(g) \in \Gamma_0$ the integral

$$\|f\|^2 = \frac{|\det g_1|^2 |\det g_2|^2}{4\pi^2} \int \left\{ \left| \frac{\partial}{\partial u} f(g_1 \sigma g_2) \right|^2 + \left| \frac{\partial}{\partial v} f(g_1 \sigma g_2) \right|^2 \right\} d\sigma \qquad (5)$$

$$(\gamma = u + iv, \qquad d\sigma = d\alpha \, d\beta \, d\gamma)$$

is finite and does not depend on g_1, g_2.

II. The linear space Γ_0 with norm (5) is mapped isomorphically and isometrically onto a certain everywhere dense in Φ linear space Φ_0, which includes the linear space $\widetilde{\Phi}$ of finite $\varphi(h_1, h_2)$, by means of the mutually inverse formulas:

$$f(g) = \int \varphi(h, h \circ g) \, dh, \qquad (6)$$

$$\varphi(h_1, h_2) = \frac{1}{4\pi^2} \int \left[L_{z_0} f\left(h_1^{-1} \hat{\zeta} \hat{z}_0 h_2 \right) \right]_{z_0 = 0} d\zeta, \qquad (7)$$

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where

$$\hat{\zeta} = \begin{Vmatrix} \zeta & 0 \\ 0 & 1 \end{Vmatrix}, \qquad \hat{z}_0 = \begin{Vmatrix} 1 & 0 \\ z_0 & 1 \end{Vmatrix}, \qquad L_{z_0} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2},$$

$$z_0 = x_0 + iy_0, \qquad d\zeta = dt\, dw, \qquad \zeta = t + iw.$$

Moreover, as was stated, $\|f\|^2 = \|\varphi\|^2$. The integral

$$\varphi(h_1, h_2) = \frac{1}{4\pi^2} |\det g_1|^2 \int \left[L_{z_0} f\left(g_1 (h_1 \circ g_1)^{-1} \hat{\zeta} \hat{z}_0 (h_2 \circ g_2^{-1}) g_2 \right) \right]_{z_0 = 0} d\zeta \quad (8)$$

does not depend on g_1, g_2, so that (8) generalizes formula (7).

III. In view of what was said in the preceding paragraph, and as formulas (7), (6) show, the function $f(g) \in \Gamma_0$ is completely determined by its values on the triangular group Σ of matrices

$$\sigma = \begin{Vmatrix} \alpha & 0 \\ \gamma & \beta \end{Vmatrix}$$

(the argument of f in (7) is triangular).

It is also completely determined by its values on any surface $g_1 \Sigma g_2$ (see (8), (6)), obtained by a two-sided shift of the group Σ.

For every infinitely smooth function $f_0(\sigma)$ on Σ, satisfying the condition

$$\left| \frac{\partial^{k_1 + k_2}}{\partial \gamma^{k_1} \partial \overline{\gamma}^{k_2}} \frac{\partial^{p_1 + p_2 + q_1 + q_2}}{\partial \alpha^{p_1} \partial \overline{\alpha}^{p_2} \partial \beta^{q_1} \partial \overline{\beta}^{q_2}} \right| < \frac{C}{(1 + |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\alpha\beta|^2)^{\frac{k_1 + k_2 + 1}{2} + \varepsilon}},$$

$$k_1 + k_2 + p_1 + p_2 + q_1 + q_2 \leq 2$$

there exists a harmonic $f(g)$ such that

$$f(\sigma) \equiv f_0(\sigma).$$

This function is given by the formula

$$f(g) = -\frac{1}{4\pi^2} \int \left\{ \int \left[L_{z_0} f_0 \left(h^{-1} \hat{\zeta} \hat{z}_0 h \circ g \right) \right]_{z_0 = 0} d\zeta \right\} dh.$$

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The question of whether $f(g)$ belongs to the space Γ_0 remains open for the present.

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Received
7 V 1963

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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