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Abstract

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SEMIGROUPS DEFINED BY IDENTITIES

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Let $\Gamma(\Phi)$ be the class of all semigroups in which the set of identities Φ is satisfied (see ([7])). The following problem naturally arises. Let $\mathfrak{H}$ be a given class of semigroups. Find all sets of identities Φ for which $\Gamma(\Phi) \subset \mathfrak{H}$. In this note a solution of this problem is given for certain important classes of semigroups $\mathfrak{H}$. Some results in this direction for classes of idempotent semigroups have been obtained in ([3–6], [9]).

Let $\Xi = \{\xi_1, \xi_2, ...\}$ be a countable alphabet; $A(\xi_i), B(\xi_i), ...$ are words (possibly empty) over the alphabet Ξ. By $l_{\xi_j}(A)$ we denote the number of occurrences of the letter ξ_j in the word $A(\xi_i)$; by $\chi(A)$, the set of those $\xi_j \in \Xi$ for which $l_{\xi_j}(A) \geq 1$; by $l(A)$, the length of the word $A(\xi_i)$. Every identity is written in the form of a formal equality of two nonempty words over the alphabet Ξ. At the same time, the identity $A(\xi_i) = B(\xi_i)$ is regarded as equal to each of the identities $\varphi A(\xi_i) = \varphi B(\xi_i)$, $\varphi B(\xi_i) = \varphi A(\xi_i)$, if φ is a one-to-one mapping of $\chi(AB)$ into Ξ. In addition, every identity of the form $A(\xi_i) = A(\xi_i)$ is regarded as equal to the empty set of identities.

A semigroup free in $\Gamma(\Phi)$ and having $\mathfrak{X}$ as its set of free generators is denoted by $F(\mathfrak{X}, \Phi)$. The semigroup $F(\mathfrak{X}, \Phi)$ is also called the semigroup defined over the alphabet $\mathfrak{X}$ by the set of identities Φ.

In most cases the question of the membership of the class of semigroups $\Gamma(\Phi)$ in a given class of semigroups $\mathfrak{H}$ reduces to the question of the membership of the semigroups $F(\mathfrak{X}, \Phi)$ (for all possible $\mathfrak{X}$) in the class $\mathfrak{H}$ (for example, this assertion, obviously, holds if the class of semigroups $\mathfrak{H}$ is closed under homomorphisms). Therefore, in solving the problem formulated above, semigroups defined by identities turn out to be the main object of investigation. Let $\Phi = \{A_\gamma(\xi_i) = B_\gamma(\xi_i),\ \gamma \in \Gamma\}$ be some set of identities. Every identity of the form $c(\xi_i) \cdot \varphi A_\gamma(\xi_i) \cdot c'(\xi_i) = c(\xi_i) \cdot \varphi B_\gamma(\xi_i) \cdot c'(\xi_i)$, where $A_\gamma(\xi_i) = B_\gamma(\xi_i)$ is some identity from Φ; φ is some mapping of $\chi(A_\gamma B_\gamma)$ into the set of nonempty words over the alphabet Ξ; and $c(\xi_i), c'(\xi_i)$ are some words (possibly empty) over the alphabet Ξ, will be called an immediate consequence of Φ. An identity $A(\xi_i) = B(\xi_i)$ will be called a consequence of Φ if there exists a finite sequence of nonempty words

$$A(\xi_i) \equiv A_1(\xi_i), \quad A_2(\xi_i), \dots, \quad A_m(\xi_i) \equiv B(\xi_i)$$

($\equiv$ is the sign of graphical equality of words) such that the identities $A_j(\xi_i) = A_{j+1}(\xi_i)$ $(j = 1, 2, \dots, m-1)$ are immediate consequences of Φ. If every identity from Ψ is a consequence of Φ, then, briefly, this is written as $\Phi \Rightarrow \Psi$.

Lemma 1. If in the semigroup $F(\mathfrak{X}, \Phi)$ $(\mathfrak{X} = \{x_\alpha, \alpha \in T\})$, $A(x_\alpha) = B(x_\alpha)$, then for any mapping $\varphi : \chi(AB) \to \Xi$ the identity $\varphi A(x_\alpha) = \varphi B(x_\alpha)$ is a consequence of Φ.

Proposition 1. In order that $\Gamma(\Phi) \subset \Gamma(\Psi)$, it is necessary and sufficient that $\Phi \Rightarrow \Psi$.

Proposition 2. Let $\mathfrak{X}$ be an infinite alphabet. In order that $\Phi \Rightarrow \Psi$, it is necessary and sufficient that $F(\mathfrak{X}, \Phi) \in \Gamma(\Psi)$.

Proposition 3. In order that $\Phi \Rightarrow \Psi$, it is necessary and sufficient that, for every finite alphabet $\mathfrak{N}$, $F(\mathfrak{N}, \Phi) \in \Gamma(\Psi)$.

For a set of identities

$$\Phi = \{A_\gamma(\xi_i) = B_\gamma(\xi_i),\ \gamma \in \Gamma\}$$

we introduce the numerical characteristic $d(\Phi)$. The greatest common divisor of the set of all positive numbers of the form

$$d(\gamma, \xi_j) = \left| l_{\xi_j}(A_\gamma) - l_{\xi_j}(B_\gamma) \right| \quad (\gamma \in \Gamma,\ \xi_j \in \Xi,\ d(\gamma, \xi_j) > 0)$$

will be denoted by $d(\Phi)$. If Φ is the empty set of identities, or if $l_{\xi_j}(A_\gamma) = l_{\xi_j}(B_\gamma)$ for all $\gamma \in \Gamma,\ \xi_j \in \Xi$, then we put $d(\Phi) = 0$.

Theorem 1. In order that the semigroup $F(\mathfrak{N}, \Phi)$ be periodic, it is necessary and sufficient that $d(\Phi) \neq 0$.

Lemma 2. If $\mathfrak{A} \in \Gamma(\Phi)$ and $\mathfrak{G}$ is some subgroup of the semigroup $\mathfrak{A}$, then for every $X \in \mathfrak{G}$

$$X^{d(\Phi)} = E_{\mathfrak{G}},$$

where $E_{\mathfrak{G}}$ is the identity of the group $\mathfrak{G}$.

We introduce several classes of sets of identities.

$\mathfrak{H}(\alpha, m)$ $(m \geq 1)$: $\Phi \in \mathfrak{H}(\alpha, m)$ if Φ contains either the identity

$$\xi_1\xi_2 \cdots \xi_t = \xi_1\xi_2 \cdots \xi_t A(\xi_i),$$

where $1 \leq t \leq m-1,\ l(A) \geq 1$, or the identity

$$\xi_1\xi_2 \cdots \xi_r\xi_{r+1} A(\xi_i) = \xi_1\xi_2\xi_3 \cdots \xi_r\xi_{r'} B(\xi_i),$$

where $0 \leq r \leq m-1,\ r' \neq r+1,\ l(A), l(B) \geq 0$.

$\mathfrak{H}^*(\alpha, m)$ is the dual class (symmetric with respect to "left" and "right") to the class of sets of identities $\mathfrak{H}(\alpha, m)$.

$\mathfrak{H}_\beta$: $\Phi \in \mathfrak{H}_\beta$ if Φ contains some identity $A(\xi_i) = B(\xi_i)$ for which $\chi(A) \neq \chi(B)$.

$\mathfrak{H}(\gamma, m)$ $(m \geq 1)$: $\Phi \in \mathfrak{H}(\gamma, m)$ if Φ contains an identity of the form

$$\xi_1 \xi_2 \cdots \xi_t = A(\xi_i),$$

where $1 \leq t \leq m$, and either $l(A) > t$, or $\chi(A) \neq \{\xi_1, \xi_2, \dots, \xi_t\}$.

$\mathfrak{H}_\delta$: $\Phi \in \mathfrak{H}_\delta$ if Φ contains an identity of the form

$$\xi_1 A(\xi_i) = B(\xi_i),$$

where $\chi(A) \neq \Phi$, $\xi_1 \in \chi(A)$, and either $l_{\xi_1}(B) \geq 2$, or $B(\xi_i) \equiv \xi_2 B'(\xi_i)$ $(l(B') \geq 0)$.

$\mathfrak{H}_\delta^*$ is the dual class of sets of identities.

$\mathfrak{H}_\varepsilon$: $\Phi \in \mathfrak{H}_\varepsilon$ if Φ contains an identity of the form

$$A_1(\xi_i)\xi_1 A_2(\xi_i) = B_1(\xi_i)\xi_2 B_2(\xi_i),$$

where $\{\xi_1, \xi_2\} \cap \chi(A_1 B_1) = \emptyset$; $l(A_1), l(A_2), l(B_1), l(B_2) \geq 0$.

$\mathfrak{H}_\varepsilon^*$ is the dual class of sets of identities.

For simplicity of formulation, some of the following theorems are not stated in their full generality.

Theorem 2. In order that the semigroup $F(\mathfrak{N}, \Phi)$ be idempotent, it is necessary and sufficient that $\Phi \in \mathfrak{H}(\gamma, 1)$, $d(\Phi) = 1$.

Theorem 3. In order that, for $m(\mathfrak{N}) \geq 2$ ($m(\mathfrak{N})$ is the cardinality of $\mathfrak{N}$), the set of all idempotents of the semigroup $F(\mathfrak{N}, \Phi)$ be a subsemigroup satisfying the identity $\xi_1 = \xi_2 \xi_1$, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}_\beta.$$

Corollary 1. In order that, for $m(\mathfrak{N}) \geq 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be a semigroup with a unique idempotent, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}_\beta.$$

Corollary 2. In order that, for $m(\mathfrak{N}) \geq 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be unitary, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, 1), \quad d(\Phi) = 1.$$

Theorem 4. In order that, for $1 \leq m \leq m(\mathfrak{N})$, $m(\mathfrak{N}) \geq 2$, the ideal $F(\mathfrak{N}, \Phi)^m$ of the semigroup $F(\mathfrak{N}, \Phi)$ be a semigroup with right cancellation, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, m).$$

Corollary 3. In order that, for $m(\mathfrak{N}) \geq 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be a group, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, 1).$$

A semigroup $\mathfrak{A}$ is called m-nilpotent (or nilpotent of class m) if $\mathfrak{A}^m = 0$, where 0 is the zero of the semigroup $\mathfrak{A}$ (see (8)).

Corollary 4. In order that, for $1 \leq m \leq m(\mathfrak{N})$, $m(\mathfrak{N}) \geq 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be m-nilpotent, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, m), \quad d(\Phi) = 1.$$

Using one lemma from ([2]), one can prove that the following is true.

Theorem 5. In order that, for $\mathfrak{m}(\mathfrak{N}) \geqslant 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be inverse, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}(\gamma, 1).$$

Theorem 6. In order that, for $\mathfrak{m}(\mathfrak{N}) \geqslant 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be completely simple without zero, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, 1).$$

Associate with the set of identities Φ the set of identities

$$\overline{\Phi} = \{\Phi,\ \xi_1 = \xi_2^{d(\Phi)}\xi_1,\ \xi_1 = \xi_1\xi_2^{d(\Phi)}\}.$$

According to Corollary 3, all semigroups in $\Gamma(\Phi)$ are groups if and only if $d(\Phi) \neq 0$. Suppose that Φ does not contain the commutativity identity (i.e. the identity $\xi_1\xi_2 = \xi_2\xi_1$). Then:

Theorem 7. 1) Let $\Phi \in \mathfrak{H}_\beta$. In order that, for $\mathfrak{m}(\mathfrak{N}) \geqslant 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be commutative, it is necessary and sufficient that

$$d(\Phi) \neq 0, \quad \Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}(\gamma, 2)$$

and that the group $F(\mathfrak{N}, \overline{\Phi})$ be commutative.

2) Let $\Phi \notin \mathfrak{H}_\beta$. In order that, for $\mathfrak{m}(\mathfrak{N}) \geqslant 2$, the semigroup $F(\mathfrak{N}, \Phi)$ be commutative, it is necessary and sufficient that

$$d(\Phi) \neq 0, \quad \Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap [\mathfrak{H}(\gamma, 1) \cup (\mathfrak{H}(\gamma, 2) \cap \mathfrak{H}_\delta \cap \mathfrak{H}^*_\delta)]$$

and that the group $F(\mathfrak{N}, \overline{\Phi})$ be commutative.

Corollary 5. 1) Let $\Phi \in \mathfrak{H}_\beta$, $d(\Phi) \leqslant 2$. In order that the semigroup $F(\mathfrak{N}, \Phi)$ be commutative, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap \mathfrak{H}(\gamma, 2).$$

2) Let $\Phi \notin \mathfrak{H}_\beta$, $d(\Phi) \leqslant 2$. In order that the semigroup $F(\mathfrak{N}, \Phi)$ be commutative, it is necessary and sufficient that

$$\Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}^*(\alpha, 1) \cap [\mathfrak{H}(\gamma, 1) \cup (\mathfrak{H}(\gamma, 2) \cap \mathfrak{H}_\delta \cap \mathfrak{H}^*_\delta)].$$

Consider identities of the form

$$\xi_1 \xi_2 \dots \xi_m = \xi_1 \xi_2 \dots \xi_r \xi_{m+1} \xi_{m+2} \dots \xi_{m+t} \xi_{m-s+1} \dots \xi_m, \tag{1}$$

where $0 \leqslant r, s \leqslant m$, $t = \max\{1, m - (s+t)\}$. It is clear that identities of the form (1) include such frequently used identities as

$$\xi_1 = \xi_2, \quad \xi_1 = \xi_2 \xi_1, \quad \xi_1 = \xi_1 \xi_2 \xi_1, \quad \xi_1 \xi_2 = \xi_3 \xi_4, \quad \xi_1 \xi_2 = \xi_1 \xi_3 \xi_2.$$

Theorem 8. In order that the identity (1) follow from the set of identities Φ, it is necessary and sufficient that

$$d(\Phi) = 1, \quad \Phi \in \mathfrak{H}(\alpha, r+1) \cap \mathfrak{H}^*(\alpha, s+1) \cap \mathfrak{H}_\beta \cap \mathfrak{H}(\gamma, m).$$

Corollary 6. If $\mathfrak{m}(\mathfrak{N}) < \infty$, $d(\Phi) = 1$, $\Phi \in \mathfrak{H}(\gamma, m) \cap \mathfrak{H}_\beta$, then the semigroup $F(\mathfrak{N}, \Phi)$ is finite.

Let

$$\Phi_1 = \{\xi_1 = \xi_1^2,\ \xi_1 \xi_2 \xi_3 \xi_4 = \xi_1 \xi_3 \xi_2 \xi_4\}, \quad \Phi_2 = \{\xi_1 = \xi_1^2,\ \xi_1 \xi_2 \xi_3 = \xi_2 \xi_1 \xi_3\}.$$

Semigroups from $\Gamma(\Phi_1)$, $\Gamma(\Phi_2)$ were used by V. V. Vagner ([1]) for the study of generalized groups.

Theorem 9. 1) Let $\Phi \in \mathfrak{H}_\beta$. In order that $\Phi \Rightarrow \Phi_1$, it is necessary and sufficient that

$$d(\Phi) = 1, \quad \Phi \in \mathfrak{H}(\gamma, 1).$$

2) Let $\Phi \notin \mathfrak{H}_\beta$. In order that $\Phi \Rightarrow \Phi_1$, it is necessary and sufficient that

$$d(\Phi) = 1, \quad \Phi \in \mathfrak{H}(\gamma, 1) \cap \mathfrak{H}_\varepsilon \cap \mathfrak{H}^*_\varepsilon.$$

Theorem 10. 1) Let $\Phi \in \mathfrak{H}_\beta$. In order that $\Phi \Rightarrow \Phi_2$, it is necessary and sufficient that

$$d(\Phi) = 1, \quad \Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}(\gamma, 1).$$

2) Let $\Phi \notin \mathfrak{H}_\beta$. In order that $\Phi \Rightarrow \Phi_2$, it is necessary and sufficient that

$$d(\Phi) = 1, \quad \Phi \in \mathfrak{H}(\alpha, 1) \cap \mathfrak{H}(\gamma, 1) \cap \mathfrak{H}_\varepsilon.$$

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Note: Figure translations are in progress. See original paper for figures.