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Abstract

Full Text

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EXTREMAL PROBLEMS FOR ANALYTIC FUNCTIONS WITH POSITIVE REAL PART SATISFYING ADDITIONAL CONDITIONS

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Mathematics

1. Let G be an n-connected domain whose boundary Γ consists of analytic contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, $\Gamma = \bigcup_1^n \Gamma_i$, and let P be the class of single-valued functions $f(z) = u(z) + iv(z)$, analytic in G, for which $u(z) > 0$ in this domain. We shall consider subclasses of the class P consisting of functions satisfying additional conditions of a definite form. Suppose, for example, that points $\zeta_1, \ldots, \zeta_m$ and $z_1, \ldots, z_k$ in the domain G, and positive numbers $A_1, \ldots, A_m$ and $B_1, \ldots, B_k$, are given. Consider the subclass P' of the class P consisting of functions for which

$$u(\zeta_j) \leq A_j, \quad j = 1, 2, \ldots, m, \quad v(z_1) = 0, \quad u(z_j) = B_j, \quad j = 1, 2, \ldots, k \tag{1}$$

(it is assumed that P' is nonempty). Instead of the conditions (1), in defining the class P' one may use more general conditions, when, for example, the set of points (ζ_j, z_j) is infinite, or linear combinations of partial derivatives of $u(z)$ at prescribed points are bounded, etc. We shall be interested in linear extremal problems in the class P'. For example, the problem of

$$\sup \operatorname{Re}\,[e^{i\theta} f(z_0)], \qquad f \in P', \tag{2}$$

where the real number θ and the point $z_0 \in G$ are fixed, or the more general problem of

$$\sup \operatorname{Re}\left[\sum \gamma_s f^{(s)}(\tau_s)\right], \qquad f \in P', \tag{3}$$

with prescribed complex numbers γ_s and points $\tau_s \in G$, etc. A special case of problems (2) and (3), when the conditions (1) reduce to the single requirement $f(\infty) = 1$, was studied by Nehari (^{1}). Linear extremal problems for bounded analytic functions with additional conditions were considered by S. Ya. Khavinson (^{2}).

The investigation of the problems posed has led us to the necessity of analyzing certain questions concerning systems of linear inequalities in normed spaces.

2. Let E be a real Banach space; let $p(x)$ be a continuous convex functional on it; let I be an arbitrary, finite or infinite, set of indices; let $X = (x_\nu;\ \nu \in I)$ be a set of elements of E; and let $(a_\nu;\ \nu \in I)$ be a set of real numbers. We shall consider linear functionals $f(x)$ on the space E satisfying the system of inequalities

$$\begin{aligned} f(x) &\leq p(x), \qquad x \in E, \\ f(x_\nu) &\geq a_\nu, \qquad \nu \in I. \end{aligned} \tag{A}$$

If there exists at least one functional f satisfying the system (A), then we shall say that the system (A) is compatible. We shall also consider linear functionals $f(x)$ on E satisfying the following system of inequalities

$$\begin{aligned} f(x) &\geq 0, \qquad x \in K, \\ f(x_\nu) &\geq a_\nu, \qquad \nu \in I, \end{aligned} \tag{B}$$

where K is a convex closed cone with nonempty interior, and in (B) the inequality

$$f(-x_0) \geq -1, \qquad x_0 \in \operatorname{int} K \tag{4}$$

holds ($\operatorname{int} \mathfrak{M}$ denotes the interior of $\mathfrak{M}$). It is easy to see that system (B) is a special case of system (A). System (A) is somewhat more general than the system considered in the work of Fan Chi (^{3}). In (^{4}) systems more general than our systems (A) and (B) were considered; however, the results obtained there are insufficient for our purposes.

Lemma 1. *System (A) is consistent if and only if, for every finite set of indices* $\nu_1, \dots, \nu_r$ *and positive numbers* $\lambda_1, \dots, \lambda_r$*, the inequality*

$$p\left(\sum_1^r \lambda_i x_{\nu_i}\right) \geq \sum_1^r \lambda_i a_{\nu_i}$$

holds (cf. (^{3}), Theorem 12, § 8).

3. Denote the shifted cone $K - x_0$ by $\widetilde{K}$, where x_0 is the same as in (4). To each $z \in K \cup X$ assign the number a, equal to a_ν if $z = x_\nu$, and equal to zero if $z \in K$. In this case it is possible that different numbers will correspond to the same element z. Introduce the sets

$$E_0 = \left\{ -\sum_1^r \lambda_i z_i : \lambda_i > 0,\ \sum_1^r \lambda_i a_i = 0 \right\}, \quad E_1 = \left\{ -\sum_1^r \lambda_i z_i : \lambda_i > 0,\ \sum_1^r \lambda_i a_i = 1 \right\},$$

$$E_2 = \left\{ \sum_1^r \lambda_i z_i : \lambda_i > 0,\ \sum_1^r \lambda_i a_i = -1 \right\},$$

where the sums are taken over arbitrary sets $z_1, z_2, \ldots, z_r$ from $K \cup X$, $r = 1, 2, \ldots$. We indicate some properties of these sets: 1) $-K \subset E_0$; $\widetilde{K} \subset E_2$; $-E_0 \subset E_2 + x_0$; 2) if $E_1 \neq \emptyset$, then $E_1 + x_0 \subset E_0$; for any $y \in E_1$ we have $E_0 + y \subset E_1$; there exists $\gamma > 0$ such that $E_2 \subset -E_0 - \gamma x_0$.

Lemma 2. *For consistency of system (B), it is necessary and sufficient that* $E_1 \cap \operatorname{int} \widetilde{K} = \emptyset$.

Lemma 3. *In order that there exist a solution* $f(x) \not\equiv 0$ *of system (B), it is necessary and sufficient that* $E_2 \neq E$.

Lemma 4. *For the existence of a solution* $f(x) \equiv 0$ *of system (B), the condition* $E_1 = \emptyset$ *is necessary and sufficient.*

Theorem 1. *If* $E_1 \neq \emptyset$, *then the set of hyperplanes separating* E_0 *and* K *coincides with the set of hyperplanes of the form* $L = \{x : f(x) = 0\}$, *determined by the solutions* $f(x)$ *of system (B).*

Theorem 2. *The set of hyperplanes separating the sets* E_1 *and* E_2 *coincides with the set of hyperplanes of the form* $L = \{x : f(x) = -1\}$, *where* $f(x)$ *is a solution of system (B).*

Let us introduce several definitions. The set $\{a_\nu\}$, $\nu \in I$, will be called **positive relative to** $\{x_\nu\}$, $\nu \in I$, if $E_0 \cap \operatorname{int} K = \emptyset$.

The set $\{a_\nu\}$ will be called **strictly positive relative to** $\{x_\nu\}$ if $E_0 \cap K = \{0\}$. In the case of the classical moment problem these definitions coincide with the usual ones (see (5)).

We shall call system (B) **regular** if, from the fact that a sequence of finite sums

$$\sum_{i=1}^{r_n} \lambda_i^{(n)} x_{\nu_i},$$

where the indices $\nu_i \in I$ may depend on n and $\lambda_i^{(n)} > 0$, converges weakly to an element $x \neq 0$ and, moreover,

$$\sum_1^{r_n} \lambda_i^{(n)} a_{\nu_i} \to 0, \quad n \to \infty,$$

it follows that the element x can be represented in the form of a finite sum:

$$x = \sum_1^r \lambda_i x_{\nu_i}, \quad \nu_i \in I, \ \lambda_i > 0 \quad \text{and} \quad \sum_1^r \lambda_i a_{\nu_i} = 0.$$

From properties 2), Lemma 2, and Theorem 1 there follows

Corollary 1. *For the consistency of system (B) it is sufficient, and if $E_1 \neq \emptyset$, also necessary, that $\{a_\nu\}$ be positive relative to $\{x_\nu\}$.*

Theorem 3. *Let the sets Q_A and Q_B of solutions of systems (A) and (B), respectively, be nonempty. Then, for every $y \in E$,*

$$\max_{f \in Q_A} f(y) = \inf \left\{ p\left(y + \sum_1^r \lambda_i x_{\nu_i} \right) - \sum_1^r \lambda_i a_{\nu_i} \right\}; \tag{5}$$

$$\max_{f \in Q_B} f(y) = \inf \left\{ -\sum_1^r \lambda_i a_{\nu_i} \right\}, \qquad -y - \sum_1^r \lambda_i x_{\nu_i} \in K, \tag{6}$$

where the lower bound in (5) and (6) is taken over all finite sets of indices $\nu_i \in I$ and numbers $\lambda_i > 0$. Moreover, the extremal functional f^* in the left-hand sides of equalities (5) and (6) always exists. In order that the functional f^* be extremal in the left-hand side of (5), and the element $y + \sum \lambda_i^* x_{\nu_i}^*$ be extremal for the right-hand side of (5), it is necessary and sufficient that

$$p\left(y + \sum \lambda_i^* x_{\nu_i}^* \right) = f^*\left(y + \sum \lambda_i^* x_{\nu_i}^* \right), \qquad f^*(x_{\nu_i}^*) = a_{\nu_i}^*.$$

For equality (6) these conditions have the form

$$f^*\left(y + \sum \lambda_i^* x_{\nu_i}^* \right) = 0; \qquad f^*(x_{\nu_i}^*) = a_{\nu_i}^*; \qquad -y - \sum \lambda_i^* x_{\nu_i}^* \in K.$$

The extremal element in (5) or (6) need not exist.

Before formulating sufficient conditions for its existence, let us give one definition. Consider the wedge

$$T = \left\{ \sum_1^r \lambda_i x_{\nu_i} : \ \lambda_i > 0, \ \nu_i \in I \right\},$$

where the sums are taken over all finite sets of indices ν_i, $r = 1, 2, \ldots$. We shall call the wedge T **regular** if $\{0\}$ does not belong to the weak closure of the set $\overline{T} \cap S$, where S is the unit sphere in E.

Theorem 4. *If the wedge T is regular, $\{a_\nu\}$ is strictly positive relative to $\{x_\nu\}$, system (B) is proper, and the space $L(X)$—the closed linear hull of X—is* *reflexive, then the extremal element* $y + \sum \lambda_i^* x_{\nu_i}^*$ *in equality (6) exists for every* $y \in E$.

From Theorem 4 there follows

Corollary 2. *If the cone* K *is pointed,* $\{a_\nu\}$ *is positive relative to* $\{x_\nu\}$*, system (B) is proper,* $L(X)$ *is reflexive, and* f^* *is the extremal functional in (6), then there exists an element* $-\delta y - \sum \lambda_i^* x_{\nu_i}^* \in K$, $\lambda_i^* > 0$, $\delta \geq 0$, *such that*

$$f^*\left(\delta y + \sum \lambda_i^* x_{\nu_i}^*\right) = 0, \qquad f^*(x_{\nu_i}^*) = a_{\nu_i}^*.$$

4. Let us return to the consideration of the extremal problem (2) in the class P' of functions defined by the conditions (1). Let $g(t, z)$ be the Green' s function of the domain G with pole at z; $h(t, z)$ its conjugate with respect to z in G; $H = (t, z) = g(t, z) + ih(t, z)$; and let $\omega_i(t)$ be the harmonic measure of the contour Γ_i. The general results set forth above, as applied to problem (1)–(2), give:

Theorem 5.

$$\sup_{f \in P'} \operatorname{Re}\left[e^{i\theta} f(z_0)\right] = \inf\left(\sum_1^m \lambda_j A_j + \sum_1^k \mu_j B_j\right), \tag{7}$$

where the infimum is taken over all possible nonnegative λ_j, $j = 1, \dots, m$, and real μ_j, $j = 1, \dots, k$, for which it is possible to choose such

real numbers $\gamma_1, \dots, \gamma_{n-1}$ such that the condition is satisfied:

$$\sin\theta \frac{\partial h(t, z_0)}{\partial n} - \cos\theta \frac{\partial g(t, z_0)}{\partial n} + \sum_1^m \lambda_j \frac{\partial g(t, \xi_j)}{\partial n} + \sum_1^k \mu_j \frac{\partial g(t, z_j)}{\partial n} + \sum_1^{n-1} \gamma_i \frac{\partial \omega_i(t)}{\partial n} \geqslant 0, \quad t \in \Gamma \tag{8}$$

($\partial/\partial n$ denotes differentiation along the inner normal to Γ).

In order that the function $f^*(z)$ be extremal in the left-hand side of (7), and the sum

$$\sum_1^m \lambda_j^* A_j + \sum_1^k \mu_j^* B_j$$

be extremal in the right-hand side, it is necessary and sufficient that, for the linear combination (8) with coefficients λ_j^*, μ_j^* and the corresponding γ_i^*, one have $\operatorname{Re} f^*(\xi_j) = A_j$ for all j for which $\lambda_j^* > 0$.

Theorem 6. If the sum

$$\sum_1^m \lambda_j^* A_j + \sum_1^k \mu_j^* B_j$$

is extremal in the right-hand side of (7), then the linear combination (8) with coefficients λ_j^*, μ_j^* and the corresponding γ_i^* has at least one zero on each contour $\Gamma_i,\ i = 1, \dots, n$. If $t_1, \dots, t_q$ are all its zeros, then the function $f^*(z)$ extremal in the left-hand side of (7) has the form

$$f^*(z) = \sum_1^{q'} \rho_j \frac{\partial H(t_j, z)}{\partial n}, \qquad q' \leqslant q,$$

$$\rho_j > 0;$$

it maps the domain G onto a q'-sheeted half-plane $\operatorname{Re} W > 0$, where $n \leqslant q' \leqslant n + m + k - 1$.

Analogous results have also been obtained for more general problems of the type (1)–(3). With the aid of these results one can also establish another criterion for sets of analytic capacity zero.

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Note: Figure translations are in progress. See original paper for figures.

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