MATHEMATICS

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1963

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Abstract

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MATHEMATICS

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SMALL PERIODIC PERTURBATIONS OF A ROUGH AUTONOMOUS SYSTEM

(Presented by Academician L. S. Pontryagin on 7 VII 1962)

In the qualitative study of nonautonomous systems it is very important to single out in phase space special manifolds separating regions of solutions with different limiting behavior. In the present note special manifolds are singled out for nonautonomous periodic systems of the second order that are close to rough autonomous systems.

Let the autonomous system

$$\frac{dx_1}{dt} = X_1(x_1, x_2), \qquad \frac{dx_2}{dt} = X_2(x_1, x_2) \tag{1}$$

be rough ([1,2]) in a region G of the plane x_1, x_2.

Consider the nonautonomous system

$$\frac{dx_1}{dt} = X_1(x_1, x_2) + \mu R_1(x_1, x_2, t); \qquad \frac{dx_2}{dt} = X_2(x_1, x_2) + \mu R_2(x_1, x_2, t), \tag{2}$$

defined in the region G_t $\{(x_1, x_2) \subset G,\ -\infty < t < +\infty\}$ of the space x_1, x_2, t. R_1 and R_2 are periodic functions of t of period τ. The functions X_1, X_2, R_1, R_2 are assumed to be functions of class C^3, and μ is a parameter.

The solutions of system (2) generate a point transformation T_τ^μ of the plane $t = t_0$ into the plane* $t = t_0 + \tau$, coinciding for $\mu = 0$ with the transformation T_τ^0 generated by the solutions of system (1).

The special trajectories ([3]) of system (1) in the plane $x_1 x_2$, by virtue of its roughness, can only be rough equilibrium states, rough limit cycles, and separatrices that do not go from saddle to saddle.

Theorem 1. *Let the rough system (1) have an equilibrium state $A(x_1^0, x_2^0)$ of the type focus or node. Then, for sufficiently small $\mu \neq 0$, system (2) has a unique isolated periodic solution Γ_μ, $(x_1 = \varphi_1(t, \mu), x_2 = \psi_1(t, \mu))$ of period τ, which for $\mu = 0$ turns into the solution $x_1 = x_1^0,\ x_2 = x_2^0$. The roots of the characteristic equations for the corresponding fixed points of the point transformations T_τ^0, T_τ^μ,* *generated by systems (1), (2), are pairwise close and are located simultaneously inside or outside the unit circle.*

The uniqueness of the periodic solution of period τ of system (2), generated by the solution $x_1 = x_1^0,\ x_2 = x_2^0$ of system (1), follows from the fact that, for the generating solution, the roots $|\lambda_i| \neq 1$ $(i = 1, 2)$. The closeness of the roots λ_i^μ and λ_i of the characteristic equations of the transformations T_τ^μ and T_τ^0 for the corresponding fixed points follows from the fact that, by continuity

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* By virtue of the periodicity in t of equations (2), the planes $t = t_0 + n\tau$ $(n = 0, \pm 1, \pm 2, ...)$ are identified.

dependence on the parameter, for sufficiently small $\mu \neq 0$ the point transformations T_τ^μ and T_τ^0 in neighborhoods of the corresponding fixed points are close [5].

Theorem 2. *Suppose that the rough system (1) has an equilibrium state $B(x_1^0, x_2^0)$ of saddle type. Then, for sufficiently small $\mu \neq 0$, system (2) has a unique isolated periodic solution Γ_μ of period τ, which for $\mu = 0$ turns into $x_1 = x_1^0,\ x_2 = x_2^0$.*

The roots of the characteristic equations of the point transformations $T_\tau^0,\ T_\tau^\mu$, generated by systems (1), (2), for the corresponding fixed points are pairwise close and are located one inside and the other outside the unit circle.

Here, from the closeness of the corresponding point transformations it follows that if $|\lambda_1| < 1,\ |\lambda_2| > 1$, then also $|\lambda_1^\mu| < 1,\ |\lambda_2^\mu| > 1$.

A fixed point of the transformation T_τ^μ for which $|\lambda_i| > 1$ or $|\lambda_i| < 1,\ i = 1, 2$ $(|\lambda_1^\mu| < 1,$ and $|\lambda_2^\mu| > 1)$ will henceforth be called a fixed point of focus or node type (a saddle fixed point).

Each cycle of the rough system (1) has one characteristic exponent different from zero.

Theorem 3. *Suppose that the rough system (1) has, on the phase plane x_1, x_2, a limit cycle l. Then, for sufficiently small $\mu \neq 0$, in the space (x_1, x_2, t) there exists a unique toroidal integral surface L^μ, filled with solutions of system (2), homeomorphic to a torus. The surface L^μ has, in the section $t = t_0$, a closed curve l^μ, which is an invariant curve of the point transformation T_τ^μ. As $\mu \to 0$, the curve l^μ contracts in the plane x_1, x_2 to the limit cycle l.*

This theorem follows as a special case for $n = 2$ from the results established by N. Levinson in [6] (see also [7–10]).

Periodic solutions of system (2) on the surface L^μ (if they exist) have [11,12] a common period $k\omega$ (k natural). In this connection, depending on the number of fixed points of the transformation $(T_\tau^\mu)^k$ (k natural), the closed curve l^μ on the surface L^μ may be filled with solutions that are periodic or almost periodic, or with a finite number of periodic solutions and nonperiodic ones tending to them.

Suppose that system (1) has an equilibrium state $B(x_1^0, x_2^0)$ of saddle type. The separatrix arcs of the saddle B, located in some neighborhood σ of it, will be denoted by s_i $(i = 1, 2, 3, 4)$. The fixed point B of the transformation T_τ^0 generates (by Theorem 2) a saddle fixed point B^μ of the transformation T_τ^μ.

Theorem 4. *For sufficiently small $\mu \neq 0$, in some neighborhood $\sigma_1 \subset \sigma$ of the saddle fixed point B^μ of the transformation T_τ^μ, there exist smooth open arcs s_i^μ $(i = 1, \dots, 4)$ adjoining the point B^μ, which are invariant arcs of the transformation T_τ^μ and contract, as $\mu \to 0$, to the parts of the separatrix arcs s_i $(i = 1, 2, 3, 4)$ lying in this neighborhood.*

Indeed, according to Hadamard' s theorem [12], in some neighborhood σ_1 of the saddle fixed point B^μ there exist two and only two invariant curves C_1^μ and C_2^μ passing through the fixed point B^μ, having a continuous [13,14] tangent. The fixed point B^μ divides the curves C_1^μ and C_2^μ into arcs s_i^μ $(i = 1, 2, 3, 4)$. It is established that, as $\mu \to 0$, the arcs s_i^μ contract in the plane x_1, x_2 to the arcs s_i.

The separatrices of the saddles of system (1), by virtue of its roughness, cannot go from saddle to saddle. Let S denote one of the separatrices of the saddle B, and s the part of it belonging to the neighborhood σ of the point B.

Theorem 5. *Suppose that the rough system (1) on the phase plane x_1, x_2 has an equilibrium state B of saddle type and a separatrix S of this saddle,*

tending as $t \to +\infty$ $(t \to -\infty)$ to the equilibrium state A of node or focus type. Then, for sufficiently small $\mu \neq 0$, in the space (x_1, x_2, t) there exists a unique integral surface Ω^μ, filled with nonperiodic solutions of system (2), tending as $t \to -\infty$ $(t \to +\infty)$ to the isolated periodic solution generated by the equilibrium state B of saddle type and, as $t \to +\infty$ $(t \to -\infty)$, to the isolated periodic solution generated by the equilibrium state A of focus or node type. The curve S^μ of the section $t = t_0$ of the surface Ω^μ is an invariant curve of the transformation T_τ^μ, going from the saddle fixed point B^μ to the fixed point A^μ of focus or node type. As $\mu \to 0$, the curve S^μ contracts in the plane x_1, x_2 to the separatrix S.

For the proof of the theorem, on the invariant separatrix arc S^μ of the saddle point B^μ (see Theorem 4) certain points are distinguished: P_0; $P_1 = T_\tau^\mu P_0$ and $P_2 = T_\tau^\mu P_1$. The totality of the preceding $(P_{-n} P_{-n+1} = T^{-n} P_0 P_1)$ and succeeding $(P_{n P_{n+1}} = T_0^{nP} P_1)$ arcs for the arc $P_0 P_1$, obtained as a result of the transformations T_τ^μ and $T_{-\tau}^\mu$ of this arc, form a continuous curve S^μ with a continuous tangent.

By virtue of the roughness of system (1) and the continuous dependence of the solutions of system (2) on the parameter and on the initial conditions, the sequences of subsequent points for the points of the curve S^μ tend to the fixed point A^μ, generated by the equilibrium state A of focus or node type. It is established that, as $\mu \to 0$, the curve S^μ contracts in the plane x_1, x_2 to the separatrix S.

Theorem 6. Let the rough system (1) in the phase plane x_1, x_2 have an equilibrium state B of saddle type and a separatrix S of this saddle, tending as $t \to +\infty$ ($t \to -\infty$) to the limit cycle l. Then, for sufficiently small $\mu \neq 0$, in the space x_1, x_2, t there exists a unique integral surface Ω^μ, filled with nonperiodic solutions of system (2), tending as $t \to -\infty$ ($t \to +\infty$) to the periodic solution generated by the equilibrium state B of saddle type and, as $t \to +\infty$ ($t \to -\infty$), to an integral surface homeomorphic to a torus, whose section $t = t_0$ (l^μ), as $\mu \to 0$, contracts in the plane x_1, x_2 to the cycle l.

The curve obtained in the section $t = t_0$ of the surface Ω^μ is an invariant curve of the point transformation T_τ^μ, contracting as $\mu \to 0$ in the plane x_1, x_2 to the separatrix S.

The proof of Theorem 5 is analogous to the proof of Theorem 4.

Definition 1. The invariant curves S^μ defined by Theorems 4 and 5 will be called **invariant separatrix curves**, and the surfaces Ω^μ−integral separatrix surfaces.

Definition 2. The fixed points, invariant closed curves, and invariant separatrix curves of the point transformation T_τ^μ will be called **special invariant curves** of the transformation T_τ^μ.

Definition 3. The integral manifolds formed in the space x_1, x_2, t by the solutions of system (2) whose initial values belong to the special invariant curves (isolated periodic solutions, toroidal and separatrix integral surfaces) will be called **special integral manifolds** of system (2).

A consequence of the theorems given above is the following

Theorem 7. Let the special trajectories of the rough system (1) partition the domain G of the plane x_1, x_2 into cells filled with nonspecial trajectories. Then, for sufficiently small $\mu \neq 0$, the special integral manifolds of sys-

systems (2) divide the domain G_t of the space $\dot{x}_1, x_2, t$ into cells filled with solutions of system (2) having the same asymptotic behavior. The cells that arise are close to the cells of system (1) interpreted in the space x_1, x_2, t, and coincide with them when $\mu = 0$.

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Received
30 VI 1962

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Note: Figure translations are in progress. See original paper for figures.