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Abstract

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HYDRODYNAMIC OSCILLATIONS OF AN INHOMOGENEOUS LOW-PRESSURE PLASMA IN A MAGNETIC FIELD

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1. Recently, much attention in the literature has been devoted to the question of the stability of an inhomogeneous low-pressure plasma confined by an external magnetic field. In solving this problem, methods are used which essentially coincide with those applied to the description of oscillations of a homogeneous plasma. A substantial shortcoming of applying such methods to the description of an inhomogeneous plasma is that, in a number of cases, they lead to a spectrum of oscillations depending on the point in space ([1]). In reality, such a result arises because the methods used are inapplicable for describing oscillations of an inhomogeneous plasma. This circumstance was pointed out in the work of V. P. Silin ([2]), in which, for the description of oscillations of a weakly inhomogeneous plasma, it was proposed to use the method of geometrical optics. This method is connected with the theory of asymptotic solutions of equations of the form

$$\lambda^2 y'' + q(\omega, x) y = 0, \tag{1}$$

where λ is a small parameter (small in comparison with the characteristic scale of variation of the function $q(\omega, x)$). The spectrum of eigenvalues in the case of real $q(\omega, x)$ is determined by the asymptotic formula

$$\oint dx\, K(\omega, x) = 2\pi n\lambda, \tag{2}$$

where $K(\omega, x)$ is the asymptotic solution of the equation

$$i\lambda K' = K^2 - q(\omega, x). \tag{3}$$

In this case, with accuracy up to terms $\sim \lambda^4$, we have ([3])

$$2\pi\lambda\left(n + \frac{1}{2}\right) =$$

$$= \oint dx \left\{ \sqrt{q} - \frac{\lambda^2}{32} q^{-5/2} \frac{\partial q}{\partial x} - \frac{\lambda^4}{2048} \left[49 q^{-11/2} \left(\frac{\partial q}{\partial x} \right)^4 - 16 q^{-7/2} \frac{\partial q}{\partial x} \frac{\partial^3 q}{\partial x^3} \right] \right\}. \tag{4}$$

Here the integration is carried out over a closed contour enclosing the turning points, but not including the singular points of the function y, lying on the real axis.

In the theory of the stability of an inhomogeneous plasma there are a number of important problems which reduce to an equation of the form (1). The present communication is devoted to the consideration of some of such problems and to the derivation of instability criteria. Let us note here that in the works ([1]), in which questions of stability of an inhomogeneous plasma were investigated, the term containing the highest derivative in equation (1) was in fact neglected. Such a procedure has a definite meaning only in those particular cases when the spectrum arising as a result proves to be independent of the coordinates.

2. Let us apply the geometrical-optics method described above to the problem of longitudinal oscillations ($\operatorname{rot} \mathbf{E} = 0,\ \mathbf{E} = -\operatorname{grad} \Phi$) of a cold magnetoactive plasma situated in a gravitational field. We shall take the magnetic field $\mathbf{B}$ to be directed along the z-axis, and the gravitational field $\mathbf{g}$ along the direction of plasma inhomogeneity—the x-axis. In this case the electrons and ions of the plasma will drift—

···along the y axis with velocities $V = -g/\omega_B$, where $\omega_B = eB/mc$. The drift velocity of the ions is M/m times greater than the velocity of the electrons.

In the most interesting case, when $v_A^2 \ll c^2$, where v_A is the Alfvén velocity (this condition will henceforth be assumed satisfied everywhere), $V^2 \ll v_A^2$, and in the frequency range $\omega \ll \omega_{Bi}$ (for longitudinal oscillations we must also require $\omega^2 \ll k_y^2 v_A^2$), the longitudinal field oscillations propagating along the drift of the particles are described by the equation

$$\Phi'' + \left(\ln \frac{N}{B^2} \right)' \Phi' + k_y^2 \left[\frac{g}{\omega \omega_1} \left(\ln \frac{N}{B} \right)' - 1 \right] \Phi = 0, \tag{5}$$

where $\omega_1 = \omega - k_{yV} i = \omega + k_y g/\omega_{Bi}$. By means of the substitution $\Phi = \frac{B}{\sqrt{N}} u$, equation (5) is reduced to the form

$$u'' + \left[k_y^2 \left\{ \frac{g}{\omega \omega_1} \left(\ln \frac{N}{B} \right)' - 1 \right\} - \frac{1}{4} \left(\ln \frac{N}{B^2} \right)'^2 - \frac{1}{2} \left(\ln \frac{N}{B^2} \right)'' \right] u = 0. \tag{6}$$

It is seen from this that the geometrical-optics method for solving equation (6) is, in any case, applicable if the inequality $k_{yL} \gg 1$ is satisfied, where L is the characteristic scale of the plasma inhomogeneity. Under this condition the last terms in the square brackets of (6) may be neglected, and for determining the oscillation spectrum one may write the relation

$$\oint dx \sqrt{k_y^2 \left[\frac{g}{\omega\omega_1}\left(\ln\frac{N}{B}\right)' - 1\right]} = 2\pi n. \tag{7}$$

To determine the frequencies of the oscillations by means of this relation it is necessary to prescribe some particular dependence $N(x)$ and $B(x)$. However, certain general conclusions about the character of the spectrum can also be drawn for quite general forms of the functions $N(x)$ and $B(x)$. Thus, in the frequency range $\omega \gg k_{yV} i$, the oscillations can become unstable ($\omega^2 < 0$) if there are regions in the plasma in which $g\left(\ln\frac{N}{B}\right)' < 0$. This condition, however, is not sufficient for instability of the oscillations. A sufficient (but not necessary) condition for instability of the oscillations is the fulfillment of this inequality throughout the plasma. In the opposite limiting case, when $\omega \ll k_{yV} i$, it is easy to see from relation (7) that the plasma oscillations are always stable. Hence we conclude that the wave vector k_y has a damping effect on the plasma oscillations. Finally, we give formulas for the spectra of oscillations in the indicated frequency ranges, respectively, for $\left(\ln\frac{N}{B}\right)' = \text{const}$ and $B\left(\ln\frac{N}{B}\right)' = \text{const}$. We have

$$\omega^2 = g\left(\ln\frac{N}{B}\right)' \left[1 + \left(\frac{\pi n}{k_y d}\right)^2\right]^{-1} \qquad \text{for } \omega \gg k_{yV} i;$$

$$\omega = \omega_{Bi}\left(\ln\frac{N}{B}\right)' k_y^{-1} \left[1 + \left(\frac{\pi n}{k_y d}\right)^2\right]^{-1} \qquad \text{for } \omega \ll k_{yV} i. \tag{8}$$

Here d is the linear size of the plasma along the x axis.

3. As a second example, let us consider oscillations of an inhomogeneous plasma confined by a strong magnetic field, when the magnetic pressure is much greater than the kinetic pressure. We shall be interested in longitudinal oscillations in the frequency range $k_z V_{Te,i} \ll \omega \ll \omega_{Bi}$ and wavelengths much greater than the Larmor radii of the electrons and ions. We direct the magnetic field along the z axis, and shall assume the plasma to be inhomogeneous only along the x axis. Under these conditions the field equations reduce to the equation

$$a_0\Phi'' + a_1\Phi' + a_2\Phi = 0, \tag{9}$$

where*

$$a_0 = \left(\frac{1}{T_i} - \frac{k_y}{\omega M \omega_{Bi}} \frac{\partial}{\partial x}\right) N T_i,$$

$$a_1 = \frac{\partial}{\partial x}\left(\frac{1}{T_i} - \frac{k_y}{\omega M \omega_{Bi}} \frac{\partial}{\partial x}\right) N T_i, \tag{10}$$

$$a_2 = -\left\{\left(k_y^2 - \frac{M}{m}\frac{\omega_{Bi}^2}{\omega^2}k_z^2\right) N - \frac{k_y}{\omega M \omega_{Bi}}\frac{\partial}{\partial x} N T_i \left(k_y^2 + \frac{M T_e}{m T_i}\frac{\omega_{Bi}^2}{\omega^2}k_z^2\right)\right\}.$$

With the substitution**

$$\Phi = u \exp\left(-\frac{1}{2}\int \frac{a_1}{a_0}\, dx\right)$$

equation (9) is reduced to the form

$$u'' + \left[\frac{a_2}{a_0} - \frac{1}{4}\left(\frac{a_1}{a_0}\right)^2 - \frac{1}{2}\left(\frac{a_1}{a_0}\right)'\right] u = 0. \tag{11}$$

In the first approximation of the geometrical-optics method, from equation (11) we obtain the following relation for determining the spectrum of plasma oscillations:

$$\oint dx \sqrt{\frac{a_2}{a_0} - \frac{1}{4}\left(\frac{a_1}{a_0}\right)^2 - \frac{1}{2}\left(\frac{a_1}{a_0}\right)'} = 2\pi n. \tag{12}$$

Let us analyze this relation in two limiting cases:

a) $\omega \gg \frac{k_{yT} i}{M \omega_{Bi}} \frac{1}{L}$ and b) $\omega \ll \frac{k_{yT} i}{M \omega_{Bi}} \frac{1}{L}$, where L is the characteristic scale of the plasma inhomogeneity.

b) For $\omega \gg \frac{k_{yT} i}{M \omega_{Bi}} \frac{1}{L}$, relation (12) has the form

$$\oint dx \sqrt{\frac{M}{m}\frac{\omega_{Bi}^2}{\omega^2}k_z^2 - k_y^2 - \frac{1}{4}(\ln N)'^2 - \frac{1}{2}(\ln N)''} = 2\pi n. \tag{13}$$

It follows from this that for $k_{yL} \gg 1$ the spectrum of oscillations of an inhomogeneous plasma coincides with the spectrum of a homogeneous plasma,

$$\omega^2 = \frac{M}{m} k_z^2 \omega_{Bi}^2 \left[k_y^2 + \left(\frac{\pi n}{d}\right)^2\right]^{-1}, \tag{14}$$

where the role of the wave number k_x is played by the quantity $\pi n/d$, where d is the plasma size along the x axis. For $k_{yL} \lesssim 1$, the spectrum of oscillations of an inhomogeneous plasma differs substantially from the spectrum of a homogeneous plasma. Moreover, if the quantity $(\ln N)'^2 + 2(\ln N)'' < 0$ in a considerable part of the plasma, then the oscillations can become unstable at sufficiently small k_y. It is easy to see that, for this, it is necessary that there be regions in the plasma in which the density increases faster than x^2.

b) In the low-frequency region $\omega \ll \dfrac{k_{yT}i}{M\omega_{Bi}}\dfrac{1}{L}$, in the expressions for a_0 and a_2 one may neglect the terms that do not contain spatial gradients. As a result, to determine the spectrum of oscillations of an inhomogeneous plasma we obtain the dispersion equation

$$\oint dx \left\{ -\frac{\left[NT_i\left(k_y^2 + \frac{MT_e}{mT_i}\frac{\omega_{Bi}^2}{\omega^2}k_z^2\right)\right]'}{(NT_i)'} - \frac{1}{4}\left(\ln(NT_i)'\right)'^2 - \frac{1}{2}\left(\ln(NT_i)'\right)'' \right\}^{1/2} = 2\pi n. \tag{15}$$

* We express our gratitude to A. B. Mikhailovskii for a valuable comment concerning the coefficient a_1.

** We shall require that $a_0 \neq 0$ throughout the plasma.

In the region of short wavelengths, when $k_y L \gg 1$, the last terms in the braces in (15) may be neglected. Then, if in the plasma $T_e/T_i = \text{const}$, from the dispersion equation (15) we find the spectrum of oscillations of an inhomogeneous plasma

$$\omega^2 = -\frac{M}{m}\frac{T_e}{T_i}k_z^2\omega_{Bi}^2\left[k_y^2 + \left(\frac{\pi n}{d}\right)^2\right]^{-1}, \tag{16}$$

from which it follows that the inhomogeneous plasma is unstable under the conditions considered. We note that these plasma oscillations may also be unstable in the case when $T_e/T_i \neq \text{const}$, but varies slowly in comparison with the variation of NT_i. We also point out that formula (16) remains valid in an arbitrary wavelength region if the ion pressure in the plasma varies linearly or as the third power of the coordinate x. In the case of an arbitrary inhomogeneity, in the wavelength region $k_y L \lesssim 1$ the last terms in the braces in (15) become, generally speaking, substantial, and to determine the spectrum of plasma oscillations it is necessary to specify the particular form of the functions $N(x)$ and $T_{e,i}(x)$. On the basis of equation (15) one can only assert that in a plasma in which $(\ln P')^2 + 2(\ln P')'' > 0$, where $P = NT_i$ is the ion pressure, and $T_e/T_i = \text{const}$, the low-frequency oscillations are always unstable.

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Note: Figure translations are in progress. See original paper for figures.