PHYSICS

V. S. KUDRYAVTSEV

1963

SovietRxiv

Abstract

Full Text

PHYSICS

V. S. KUDRYAVTSEV

ON THE THEORY OF THE MOTION OF CHARGED PARTICLES IN A STRONG MAGNETIC FIELD IN THE PRESENCE OF A STRONG TRANSVERSE ELECTRIC FIELD

(Presented by Academician M. A. Leontovich, 2 IV 1963)

1. The equations of motion of a charged particle in electric and magnetic fields in the nonrelativistic case have the form:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{1}$$

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\mathbf{E} + \frac{e}{mc}[\mathbf{vB}], \tag{2}$$

where $\mathbf{r}, \mathbf{v}$ are the radius vector and velocity of the particle; e, m are the charge and mass of the particle; c is the speed of light; $\mathbf{E}, \mathbf{B}$ are the electric and magnetic fields. In the works of N. N. Bogolyubov and D. N. Zubarev ([1]) and S. I. Braginskii ([2]), equations of the averaged motion of a charged particle in a strong magnetic field were obtained by a systematic expansion.

The small parameter of the expansion in the cited works is the ratio of the Larmor radius to the distance L over which the magnetic field changes appreciably, i.e. the quantity $\varepsilon = \frac{mc^2}{eBL}\frac{v}{c}$. (In the nonrelativistic approximation $\frac{v}{c}$ is itself small; therefore, for small ε the factor $\frac{mc^2}{eBL}$ need not be small.) In addition, it is assumed that the ratio of the velocity of the electric drift $\frac{cE_\perp}{B}$ to the particle velocity v, equal to the ratio of the first and second terms on the right-hand side of equation (2), is of order ε, or that the transverse electric field $E_\perp$ must be sufficiently weak:

$$\frac{E_\perp}{B} \sim \varepsilon\frac{v}{c} \sim \frac{mc^2}{eBL}\left(\frac{v}{c}\right)^2.$$

In this case the displacement of the Larmor circle during one revolution is small in comparison with the Larmor radius. In the present work the restriction of the smallness of the electric-drift velocity $\frac{cE_\perp}{B}$ relative to the particle velocity v is removed; they may have the same order of magnitude, while the motion across the magnetic field in the zeroth approximation will already occur not along a circle, but along a trochoid. Let us separate in equation (1) the part of the drift motion caused by the electric field:

$$\frac{d\mathbf{r}}{dt} = c\frac{[\mathbf{EB}]}{B^2} + \mathbf{v}', \tag{3}$$

where both terms on the right-hand side of (3) have the same order of magnitude. The component of the electric field perpendicular to the magnetic field, $E_\perp$, in our case is increased by a factor $1/\varepsilon$ in comparison with the former value E, so that

$$\frac{E_\perp}{B} \sim \frac{v}{c}.$$

The order of the component $E_\parallel$ along the magnetic field remains the same as before,

$$\frac{E_\parallel}{B} \sim \varepsilon\,\frac{v}{c} \sim \frac{mc^2}{eBL}\left(\frac{v}{c}\right)^2.$$

Thus, equation (2) has the form

$$\frac{d\mathbf{v}}{dt} = \mathbf{E}_\parallel + \frac{\mathbf{E}_\perp}{\varepsilon} + \frac{[\mathbf{vB}]}{\varepsilon}, \tag{4}$$

where here and below in the equations we shall set equal to unity constant numerical coefficients depending on the choice of units of measurement.

2. To bring the equations of motion to a form convenient for eliminating the rapid motion, let us find the expression for the classical Poisson brackets for two quantities f, g, depending on $\mathbf{v}$ and $\mathbf{r}$, with $\mathbf{p} = \mathbf{v} + \mathbf{A}$, where $\mathbf{A}(\mathbf{r})$ is the vector potential of the magnetic field. Taking into account that $\operatorname{rot}\mathbf{A} = \mathbf{B}$, we obtain

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{v}}\frac{\partial g}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}}\frac{\partial g}{\partial \mathbf{v}} - \mathbf{B}\left[\frac{\partial f}{\partial \mathbf{v}}\frac{\partial g}{\partial \mathbf{v}}\right]. \tag{5}$$

Let the unit vector in the direction of the magnetic field be equal to $\vec{\tau}_0$, and let two mutually perpendicular unit vectors perpendicular to $\vec{\tau}_0$ be equal to $\vec{\tau}_1$ and $\vec{\tau}_2$.

We decompose the velocity $\mathbf{v}'$ in equation (3) along the trihedron $\vec{\tau}_0, \vec{\tau}_1, \vec{\tau}_2$; then for the total velocity we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{V} + v_\parallel \vec{\tau}_0 + v_\perp \vec{\tau}_1 \cos\alpha - v_\perp \vec{\tau}_2 \sin\alpha, \tag{6}$$

where

$$\mathbf{V} = \frac{[\mathbf{E}\vec{\tau}_0]}{B}, \qquad v_\parallel = \mathbf{v}\vec{\tau}_0, \qquad v_\perp = \sqrt{(\mathbf{v}-\mathbf{V})^2 - v_\parallel^2},$$

$$\cos\alpha = \frac{1}{v_\perp}(\mathbf{v}-\mathbf{V}, \vec{\tau}_1), \qquad \sin\alpha = -\frac{1}{v_\perp}(\mathbf{v}-\mathbf{V}, \vec{\tau}_2).$$

From this we obtain the expressions for the Poisson brackets:

$$\{v_\perp, \alpha\} = \frac{B}{v_\perp} + \frac{v_\parallel}{v_\perp}\vec{\tau}_0 \operatorname{rot}\vec{\tau}_0 + \frac{\vec{\tau}_0}{v_\perp}\operatorname{rot}\mathbf{V} - \vec{\tau}_1(\vec{\tau}_1\nabla)\vec{\tau}_2\cos\alpha - \vec{\tau}_2(\vec{\tau}_2\nabla)\vec{\tau}_1\sin\alpha,$$

$$\{v_\parallel, \alpha\} = -\frac{1}{2}\vec{\tau}_0\operatorname{rot}\vec{\tau}_0 + \vec{\tau}_2(\vec{\tau}_0\nabla)\vec{\tau}_1 + \left(\frac{v_\parallel}{v_\perp}\vec{\tau}_2(\vec{\tau}_0\nabla)\vec{\tau}_0 - \frac{\vec{\tau}_1}{v_\perp}\operatorname{rot}\mathbf{V}\right)\cos\alpha +$$

$$+\left(\frac{v_\parallel}{v_\perp}\vec{\tau}_1(\vec{\tau}_0\nabla)\vec{\tau}_0 + \frac{\vec{\tau}_2}{v_\perp}\operatorname{rot}\mathbf{V}\right)\sin\alpha + \frac{1}{2}\left(\vec{\tau}_1(\vec{\tau}_2\nabla)\vec{\tau}_0 + \vec{\tau}_2(\vec{\tau}_1\nabla)\vec{\tau}_0\right)\cos 2\alpha +$$

$$+\frac{1}{2}\left(\vec{\tau}_1(\vec{\tau}_1\nabla)\vec{\tau}_0 - \vec{\tau}_2(\vec{\tau}_2\nabla)\vec{\tau}_0\right)\sin 2\alpha,$$

$$\{v_\perp, v_\parallel\} = \frac{v_\perp}{2}\operatorname{div}\vec{\tau}_0 + \left(v_\parallel\vec{\tau}_1(\vec{\tau}_0\nabla)\vec{\tau}_0 + \vec{\tau}_2\operatorname{rot}\mathbf{V}\right)\cos\alpha + \left(-v_\parallel\vec{\tau}_2(\vec{\tau}_0\nabla)\vec{\tau}_0 +\right.$$

$$\left.+\vec{\tau}_1\operatorname{rot}\mathbf{V}\right)\sin\alpha + \frac{v_\perp}{2}\left(\vec{\tau}_1(\vec{\tau}_1\nabla)\vec{\tau}_0 - \vec{\tau}_2(\vec{\tau}_2\nabla)\vec{\tau}_0\right)\cos 2\alpha -$$

$$-\frac{v_\perp}{2}\left(\vec{\tau}_1(\vec{\tau}_2\nabla)\vec{\tau}_0 + \vec{\tau}_2(\vec{\tau}_1\nabla)\vec{\tau}_0\right)\sin 2\alpha, \tag{7}$$

$$\{v_\perp, \mathbf{r}\} = \vec{\tau}_1\cos\alpha - \vec{\tau}_2\sin\alpha,$$

$$\{v_\parallel, \mathbf{r}\} = \vec{\tau}_0.$$

$$\{\mathbf{r}, \alpha\} = \frac{1}{v_\perp}(\vec{\tau}_2 \cos\alpha + \vec{\tau}_1 \sin\alpha),$$

$$\{x_i, x_k\} = 0.$$

The expression for the Hamiltonian function is obtained from (6):

$$\mathscr{H} = \frac{1}{2}v^2 + \varphi = \frac{1}{2}V^2 + v_\perp(\mathbf{V}\vec{\tau}_1 \cos\alpha - \mathbf{V}\vec{\tau}_2 \sin\alpha) + \frac{1}{2}v_\perp^2 + \frac{1}{2}v_\parallel^2 + \varphi, \quad (8)$$

where $\varphi(\mathbf{r})$ is the scalar potential of the electric field. With the aid of the Hamiltonian and the Poisson brackets one can now obtain the derivatives with respect to

time on $\mathbf{r}, v_\parallel, v_\perp$, and α, since the time derivative is equal to the Poisson bracket of the Hamiltonian and the given quantity. In this way we again obtain formula (6). The equations for the remaining quantities have a more cumbersome form than equations (7), and $\mathscr{H}$ is an obvious integral of these equations.

The equations of the averaged motion are obtained by changing variables so that the expressions for the Poisson brackets in the new variables do not contain the new cyclic variable. Then the cyclic coordinate will also be eliminated from the equations of motion.

3. We shall denote the new variables by a bar above them, and seek the required change of variables in the form of an expansion in inverse powers of the large quantity B: $v_\perp = \bar{v}_\perp + v_{\perp 1}(\bar{v}_\perp, \bar{\alpha}, \bar{v}_\parallel, \bar{\mathbf{r}}) + v_{\perp 2} + \cdots$, and analogously for the remaining variables. We take the mean values with respect to $\bar{\alpha}$ of the small terms to be zero. Substituting the new variables into (7) and equating separately the terms of the same order that oscillate and those that do not depend on $\bar{\alpha}$, we find $v_{\parallel 1}$, $\mathbf{r}_1$, the combination of the derivatives of $\alpha_1, v_{\perp 1}$, and the zero-order terms for the Poisson brackets of the new variables. The Hamiltonian $\mathscr{H}_0$ in the zero approximation is equal to (we omit the bar over the new variables)

$$\mathscr{H}_0 = \frac{1}{2}V^2 + \frac{1}{2}v_\perp^2 + \frac{1}{2}v_\parallel^2 + \varphi. \quad (9)$$

4. In the first approximation $\mathscr{H}_1$ from (8) contains the incompletely determined quantities $v_{\perp 1}, \alpha_1$ and depends on α. By the choice of $v_{\perp 1}$ and α_1 one can achieve the elimination of α from $\mathscr{H}_1$. The quantity $\mathbf{r}_2$, which enters into $\mathscr{H}_1$ owing to the large value of the last term in (8), is found analogously to the preceding one. Thus:

$$\mathscr{H}_1 = -\frac{V^2}{B}\vec{\tau}_0 \operatorname{rot}\mathbf{V} + \frac{1}{2B}[\mathbf{V}\vec{\tau}_0]\nabla V^2 + \frac{v_\parallel}{B}\mathbf{V}\operatorname{rot}\mathbf{V} - \frac{v_\parallel}{B}V^2\vec{\tau}_0\operatorname{rot}\vec{\tau}_0 \\ + \frac{v_\parallel^2}{B}\mathbf{V}\operatorname{rot}\vec{\tau}_0 + \frac{3}{4B^2}v_\perp^2[\mathbf{V}\vec{\tau}_0]\nabla B - \frac{v_\perp^2}{4B}\vec{\tau}_0\operatorname{rot}\mathbf{V}. \tag{10}$$

The indicated process of eliminating α from $\mathscr{H}$ and from the Poisson brackets can be continued further. This situation to some extent recalls perturbation theory in the presence of expression (3). The expressions for the Poisson brackets in the new variables must satisfy the relations following from the Jacobi identity (4). It can be checked that in our case these relations are fulfilled.

5. The derivatives of the new variables are equal to the Poisson brackets of the Hamiltonian (9) and (10) with the corresponding new variable. Thus the averaged drift equations have the form

$$\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \mathbf{V} + v_\parallel\vec{\tau}_0 + \frac{v_\perp^2}{2B}\vec{\tau}_0(\vec{\tau}_0\operatorname{rot}\vec{\tau}_0) + \frac{v_\perp^2}{2B^2}[\vec{\tau}_0\nabla B] + \frac{v_\parallel^2}{B}[\vec{\tau}_0(\vec{\tau}_0\nabla)\vec{\tau}_0] \\
&\quad + \frac{v_\parallel}{B}\operatorname{rot}\mathbf{V} + \frac{v_\parallel\vec{\tau}_0}{B}(\vec{\tau}_0\operatorname{rot}\mathbf{V}) - \frac{v_\parallel\mathbf{V}}{B}(\vec{\tau}_0\operatorname{rot}\vec{\tau}_0) + \frac{1}{B}[\vec{\tau}_0(\mathbf{V}\nabla)\mathbf{V}], \\
\frac{dv_\parallel}{dt} &= \frac{v_\perp^2}{2}\operatorname{div}\vec{\tau}_0 + \mathbf{E}\vec{\tau}_0 - (\vec{\tau}_0\nabla)\frac{V^2}{2} + \frac{v_\parallel}{B}(\mathbf{E}\operatorname{rot}\vec{\tau}_0) + \frac{\mathbf{E}\operatorname{rot}\mathbf{V}}{B}, \\
\frac{dv_\perp}{dt} &= -\frac{v_\parallel v_\perp}{2}\operatorname{div}\vec{\tau}_0 + \frac{v_\perp}{2B^2}\mathbf{E}[\vec{\tau}_0\nabla B],
\end{aligned} \tag{11}$$

$$\frac{d\alpha}{dt} = B + \frac{[\mathbf{V}\vec{\tau}_0]}{B}\nabla B + \frac{1}{2}\vec{\tau}_0\operatorname{rot}\mathbf{V} + 2v_\parallel\vec{\tau}_2(\vec{\tau}_0\nabla)\vec{\tau}_1 + \frac{3}{2}\frac{\mathbf{E}\nabla B}{B^2} \\ + \frac{\mathbf{E}(\vec{\tau}_1\nabla)\vec{\tau}_1}{B} + \frac{\mathbf{E}(\vec{\tau}_2\nabla)\vec{\tau}_2}{B}.$$

Averaged equations of type (11) could also have been obtained by direct averaging of the initial equations of motion without invoking the canonical formalism; however, in that case it is difficult to discern the integral relations that follow from the averaged equations with accuracy up to second order.

6. Along with the Hamiltonian, an integral of the averaged drift equations is the adiabatic invariant, the M-momentum canonically conjugate to the excluded cyclic coordinate a; this means that

$$\{M, a\} = 1, \qquad \{\mathscr{H}, M\} = 0. \tag{12}$$

Solving these equations, we find, to accuracy B^{-3} (in ordinary units),

$$M = \frac{mv_\perp^2}{2B} - \frac{m^2c}{e}\frac{v_\parallel v_\perp^2}{2B^4}\,\mathbf{B}\,\mathrm{rot}\,\mathbf{B} + \frac{m^2c^2}{e}\frac{v_\perp^2}{2B^3}\,\mathbf{B}\,\mathrm{rot}\,\frac{[\mathbf{EB}]}{B^2}. \tag{13}$$

The ratio of the last term, due to the transverse electric field in (13), to the first term (for $v_\parallel = 0$) is of the order of

$$\frac{mc^2}{eBL}\frac{E}{B}.$$

In a trap with magnetic mirrors, described in (5), with parameters $L = 25$ cm, $B = 5000$ oersteds, $\varphi_L - \varphi_l = 30$ kV, $l = 0.5$ cm (the radius of the axial plasma beam), this ratio reaches 0.26; however, in the case when the radial electric field E has axial symmetry, the curl in the last term in (13) vanishes.

In conclusion, I express my gratitude to S. I. Braginskii for his attention to this work.

Received
20 III 1963

REFERENCES

[1] N. N. Bogoliubov, D. N. Zubarev, *Ukr. Mat. Zh.*, **7**, No. 1, 5 (1955); N. N. Bogoliubov, Yu. A. Mitropolskii, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1958.
[2] S. I. Braginskii, *Ukr. Mat. Zh.*, **8**, No. 2, 119 (1956).
[3] M. Born, *Vorlesungen über Atommechanik*, 1, 1925, § 43.
[4] L. D. Landau, E. M. Lifshitz, *Mechanics*, Moscow, 1958, § 42.
[5] M. S. Ioffe, R. I. Sobolev, V. G. Telkovskii, E. E. Yushmanov, *ZhETF*, **39**, 1602 (1960).

Note: Figure translations are in progress. See original paper for figures.