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Abstract

Full Text

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On the Cardinalities of Sets of Precomplete Classes of Certain Functional Systems Related to Automata

(Presented by Academician P. S. Novikov on 21 I 1963)

This note considers questions related to the completeness of certain functional systems describing properties of automata. Similar systems of a special kind were studied earlier in ^(1,2). It is known that completeness conditions can be formulated in terms of precomplete classes ⁽³⁾, and the effectiveness of these conditions depends on the cardinality of the set of the indicated classes. Below it is shown that, under a certain natural choice of operations (even stronger as applied to automata than those usually considered ^(1,2,4,5)), the cardinality of the set of all precomplete classes of o.d. (boundedly deterministic) operators ⁽⁴⁾ is equal to the continuum, while the analogous cardinality of deterministic operators ⁽⁴⁾ is equal to the hypercontinuum. It is interesting to note that the first fact holds although the set of all o.d. operators is countable and has a finite basis.

1°. Consider mappings of sequences

$$x(1), x(2), \dots, x(t), \dots \tag{1}$$

into sequences

$$y(1), y(2), \dots, y(t), \dots, \tag{2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $y(t) = (y_1(t), y_2(t), \dots, y_m(t))$, $n \geq 1$. The variables $x_i(t)$, $y_j(t)$, $i = 1, \dots, n$, $j = 1, \dots, m$, hereinafter called respectively **input** and **output variables** (or, more briefly, inputs and outputs), may assume only the two values 0 or 1. Let such a mapping T be given by the system

$$\begin{aligned} q(1) &= q_0, \\ q(t+1) &= \psi[q(t), x_1(t), \dots, x_n(t)], \\ y_1(t) &= \varphi_1[q(t), x_1(t), \dots, x_n(t)], \\ &\dots \dots \dots \\ y_m(t) &= \varphi_m[q(t), x_1(t), \dots, x_n(t)], \end{aligned} \tag{3}$$

where the range of values of $q(t)$ is finite; then T is called an o.d. operator. We shall denote it by $T(x_1, \dots, x_n) = (y_1, \dots, y_m)$. Obviously, one and the same mapping T can be given by different systems of the form (3). It may happen that in (3) each of the functions $\psi, \varphi_j, 1 \leq j \leq m$, depends inessentially on the variables $x_1(t), \dots, x_n(t)$. In this case the mapping T is called **constant**, and by definition it is assumed that T maps any sequence of the form (1) into some fixed sequence. Let us describe several operations on o.d. operators. Suppose there are an input alphabet $X = \{x_1, x_2, \dots, x_n, \dots\}$ and an output alphabet $Y = \{y_1, y_2, \dots, y_m, \dots\}$.

A. Let a mapping T be given by the system (3). Discarding from (3) all equations except the first two and the equation $y_i(t) = \varphi_i[q(t), x_1(t), \dots, x_n(t)]$, we obtain a new system. The mapping of the sequence (1) into (2), where $x(t) = (x_1(t), x_2(t), \dots, x_n(t)), y(t) = (y_1(t), \dots, y_m(t)), 1 \leq i \leq m$, given by the new system, will be denoted by $T_{(i)}(x_1, \dots, x_n) = y_i$. By definition, $T_{(i)}$ is obtained from T by operation A.

B. Let $x_k \in X, y_s \in Y, s > m$, and let the mapping

$$T(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = (y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m); \quad (*)$$

then the operator

$$T(x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, x_n) = (y_1, \dots, y_{j-1}, y_s, y_{j+1}, \dots, y_m)$$

is, by definition, obtained from (*) by operation B.

V. Suppose we have the mappings

$$T_1(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) = (y_1, \dots, y_m),$$

$$T_2(x_{n+1}, \dots, x_{n+p}) = (y_{m+1}, \dots, y_{m+s}).$$

Then the operator

$$T(x_1, \dots, x_{k-1}, T_{2(u)}(x_{n+1}, \dots, x_{n+p}), x_{k+1}, \dots, x_n) = (y_1, \dots, y_m),$$

$1 \leq k \leq n, m+1 \leq u \leq m+s$, is, by definition, obtained from T_1 and T_2 by operation V.

Fig. 1

Fig. 1

G. Suppose we have the mappings

$$T_1(x_1, \dots, x_n) = (y_1, \dots, y_m),$$

$$T_2(x_{n+1}, \dots, x_{n+p}) = (y_{m+1}, \dots, y_{m+s}).$$

Then the operator that maps the sequence (1) into (2), where

$$x(t) = (x_1(t), \dots, x_n(t), x_{n+1}(t), \dots, x_{n+p}(t)),$$

$$y(t) = (y_1(t), \dots, y_m(t), y_{m+1}(t), \dots, y_{m+s}(t)),$$

and that coincides with T_1 in the computation of $y_k(t)$, $1 \leq k \leq m$, and with T_2 in the computation of $y_w(t)$, $m + 1 \leq w \leq m + s$, is, by definition, obtained from T_1 and T_2 by operation (2).

D. Let the mapping T be given by system (3). Suppose that, for some k ,

$$y_k(t) = \varphi_k[q(t), x_1(t), \dots, x_{i-1}(t), x_{i+1}(t), \dots, x_n(t)]$$

holds (obviously, any other representation of T in the form (3) will have this property). Substituting everywhere in (3), instead of $x_i(t)$,

$$\varphi_k(q(t), x_1(t), \dots, x_{i-1}(t), x_{i+1}(t), \dots, x_n(t)),$$

we obtain a new system and consider the mapping T' of sequences (1) into (2), where

$$x(t) = (x_1(t), \dots, x_{i-1}(t), x_{i+1}(t), \dots, x_n(t)),$$

$$y(t) = (y_1(t), \dots, y_{k-1}(t), y_{k+1}(t), \dots, y_m(t)),$$

given by the new system. By definition, T' is obtained from T by operation D (in the language of automata this operation is called “feedback”).

It can be shown that the application of any of the operations described to finite-valued deterministic operators yields a finite-valued deterministic operator. In Fig. 1 the application of operations A, B, V, G, D to automata is illustrated. Denote by P the set of all finite-valued deterministic operators, and by \mathfrak{M} any subset of P . The set \mathfrak{M} is called **closed** if it is closed with respect to the operations A, B, V, G, D. The set $[\mathfrak{M}]$ is called the **closure** of \mathfrak{M} if it contains those and only those operators that are obtained from the operators of the set \mathfrak{M} by a finite number of applications of the operations A, B, V, G, D. Obviously, if \mathfrak{M} is closed, then $\mathfrak{M} = [\mathfrak{M}]$. \mathfrak{M} is called **complete** if $[\mathfrak{M}] = P$. \mathfrak{M} forms a **precomplete class** if $[\mathfrak{M}] \neq P$, but

$$[\mathfrak{M} \cup \{T\}] = P$$

for every finite-valued deterministic operator $T \notin \mathfrak{M}$. Obviously, a precomplete class is closed. Let T be given by system (3). The tuple of values $(a_1(t), \dots, a_n(t))$ of the input variables $x_1(t), \dots, x_n(t)$ will be called an **input letter** and denoted by $a(t)$. An **input word** will be called a sequence—

ness of the form

$$a(1), a(2), \dots, a(t), \dots; \tag{4}$$

analogously, the notion of an **output letter** $b(t)$ and an **output word**

$$b(1), b(2), \dots, b(t), \dots \tag{5}$$

is introduced.

It is known (4) that a constant O.D. operator transforms any input word into one and the same periodic output word. Let C be the set of all constant O.D. operators with one output. An operator from C with output word α shall be denoted by $\Gamma(\alpha)$; by $]\Gamma(\alpha)[$ we shall denote the set of all operators from C whose output words differ from α in a finite number of output letters. Let

$$A = \bigcup_{i \in I} \alpha_i,$$

where α_i is a periodic sequence of 0' s and 1' s, and I is some set of natural numbers. Introduce the notation:

$$V_A = \bigcup_{\alpha_i \in A} \Gamma(\alpha_i), \quad]V_A[= \bigcup_{\alpha_i \in A}]\Gamma(\alpha_i)[.$$

Let $T(x_1, \dots, x_n) = (y_1, \dots, y_m)$ from P be given by system (3), and let $C' \subset C$. Consider the O.D. operator

$$T(T_{i_1}, T_{i_2}, \dots, T_{i_n}) = (y_1, \dots, y_m),$$

where $T_{i_1}, T_{i_2}, \dots, T_{i_n} \in C'$. Obviously, it is constant. We shall say that T preserves C' if, for any $T_{ij} \in C'$, $j = 1, \dots, n$, $i = 1, 2, \dots$, and any k , $1 \leq k \leq m$, $T_{(k)}(T_{i_1}, \dots, T_{i_n}) = y_k$ belongs to C' , and \mathfrak{M} preserves $C' \subset C$ if every operator from \mathfrak{M} preserves C' . Let \mathfrak{M} preserve $C' \subset C$. We shall call a set $\mathfrak{M}^* \subset P$ C' -maximal for \mathfrak{M} if \mathfrak{M}^* is closed, preserves C' , contains \mathfrak{M} , and $[\mathfrak{M}^* \cup \{T\}]$ does not preserve C' for any $T \notin \mathfrak{M}^*$.

Fig. 2

Lemma 1. For any $\mathfrak{M} =]\mathfrak{M}[$ and C' such that \mathfrak{M} preserves C' , the set \mathfrak{M}^* exists.

Let

$$\tilde{C} = \bigcup_{i=1}^{\infty} \Gamma(\beta_i),$$

where

$$\beta_i = \underbrace{0 \dots 0}_i 1 \underbrace{0 \dots 0 1 0 \dots}_i.$$

Lemma 2. Let $\tilde{C}' \subset \tilde{C}$, $\tilde{C}'' \subset \tilde{C}$, $\tilde{C}' \neq \tilde{C}''$; then

$$]\tilde{C}'[\neq]\tilde{C}''[$$

and

$$||\tilde{C}'|| \neq ||\tilde{C}''||.$$

Let δ be a periodic sequence of zeros and ones

$$\delta_1, \delta_2, \dots, \delta_t, \dots, \quad (6)$$

such that $\Gamma(\delta) \in]\tilde{C}'[\subset \tilde{C}$. Consider an O.D. operator $T(x_1, \dots, x_n) = y_1$ having the following property. For $T(x_1, \dots, x_n) = y_1$ there exist an input $x_k(t)$, one and the same for any sequence (6), and a number $t_0 = t_0(\delta, T)$, such that any input word of the form (4), where $a_k(t) = \delta_t$ for $1 \leq t \leq t_0$, is transformed by T into an output word (5) such that $b(t) = a_k(t)$ for $t > t_0$. In this case we shall say that the input $x_k(t)$ of the operator T has the **F-property** on the set $]\tilde{C}'[$. As is not hard to show, an O.D. operator some input of which has the **F-property** on $]\tilde{C}'[$ preserves $]\tilde{C}'[$. An O.D. operator $T(x_1, \dots, x_n) = y_1$ is called **almost constant** if there exists a number $t_1 = t_1(T)$ such that T transforms any pair of input words into output words of the form (5) and, respectively,

$$b'(1), b'(2), \dots, b'(t), \dots \quad (7)$$

so that $b(t) = b'(t)$ for $t \geq t_1$. Suppose we have an O.D. operator

$$T(x_1, \dots, x_n) = (y_1, \dots, y_m)$$

such that each of the operators $T_{(k)}(x_1, \dots, x_n) = y_k$, $1 \leq k \leq m$, is either almost constant and preserves $]\tilde{C}'[$, or has

input possessing the **F-property** on $]\tilde{C}'[$. The set of all such o.d. operators will be denoted by $\mathfrak{N}(\tilde{C}')$.

Remark 1. $\mathfrak{N}(\tilde{C}')$ preserves $]\tilde{C}'[$.

Remark 2. The set of constant o.d. operators in $\mathfrak{N}(\tilde{C}')$ coincides with $[[\tilde{C}'[$.

In Fig. 2 an automaton is shown that realizes the o.d. operator $T_{\beta, Q}(x_1, \dots, x_n) = y_1$, where $f_1(u_1, x_1) = u_1 \sim x_1$, $f_2(u_2, u_3) = u_2 \cdot u_3$, $f_4(u_4, x_1, y_1) = \bar{u}_4 x_1 \vee u_4 y_1$, the automaton A_β realizes the o.d. operator $\Gamma(\beta)$, f_3 is a unit delay with initial output state 1, and A_Q realizes an arbitrary fixed o.d. operator $Q(x_2, \dots, x_n) = y_1$ with one output. Obviously, the input $x_1(t)$ of the operator $T_{\beta, Q}$ possesses the **F-property** on $]\tilde{C}'[$ if $\Gamma(\beta)$ (see Fig. 2)

does not belong to $]\tilde{C}'[$; hence $T_{\beta,Q} \in \mathfrak{N}(\tilde{C}')$. Moreover, it is easy to see that $T_{\beta,Q}(\Gamma(\beta), x_2, \dots, x_n) = y_1$ coincides with $Q(x_2, \dots, x_n) = y_1$.

Lemma 3. *The set $\mathfrak{N}(\tilde{C}')$ is closed.*

By Lemma 1 there exists $]\tilde{C}'[$ —the maximal set $\mathfrak{N}^*(\tilde{C}')$ for $\mathfrak{N}(\tilde{C}')$.

Lemma 4. *Every set $\mathfrak{N}^*(\tilde{C}')$ forms a precomplete class.*

Proof. We shall show that if $T \notin \mathfrak{N}^*(\tilde{C}')$ and Q is any o.d. operator with one output, then $Q \in \mathfrak{M} = [\mathfrak{N}^*(\tilde{C}') \cup \{T\}]$. Indeed, by the definition of the set $\mathfrak{N}^*(\tilde{C}')$, the set \mathfrak{M} contains an o.d. operator $T'(x_{i_1}, \dots, x_{i_n}) = (y_{j_1}, \dots, y_{j_m})$ such that the constant o.d. operator $T'_{(j_k)}(T_{z_1}, T_{z_2}, \dots, T_{z_n}) = y_{j_k}$, for some $T_{z_l} \in]\tilde{C}'[$ and $k \leq m$, where $l \leq n$, does not belong to $]\tilde{C}'[$; let its output word be β . As indicated above, $\mathfrak{N}(\tilde{C}')$ contains an o.d. operator $T_{\beta,Q}(x_1, \dots, x_n) = y_1$ such that $T_{\beta,Q}(\Gamma(\beta), x_2, \dots, x_n) = y_1$ coincides with Q . It is not difficult to show that any o.d. operator can be constructed from o.d. operators with one output, using the operations B and Γ .

Theorem 1. *The cardinality \mathfrak{m} of the set of all precomplete classes of operators is equal to the continuum (\mathfrak{c}).*

Proof. Obviously, $\mathfrak{m} \leq \mathfrak{c}$. Let $\tilde{C}' \subset \tilde{C}$, $\tilde{C}'' \subset \tilde{C}$, $\tilde{C}' \neq \tilde{C}''$. By Lemma 2, $]\tilde{C}'[\neq]\tilde{C}''[$; hence, by Remark 2, $\mathfrak{N}(\tilde{C}') \neq \mathfrak{N}(\tilde{C}'')$, i.e. $\mathfrak{N}^*(\tilde{C}') \neq \mathfrak{N}^*(\tilde{C}'')$ (as sets preserving different sets), while there are continuum many distinct subsets in \tilde{C} .

Using constructions somewhat different from those considered, one can prove an analogous assertion in some other cases as well, for example, if in defining feedback one requires only the presence of “non-contradictoriness” (^{4,6}), or if the initial state of the operators is not regarded as fixed.

2°. Consider operators specified by systems of the form (3) and such that the range of values of $q(t)$ is finite or countable (deterministic operators (⁴)). Then in all the cases of operations described above, the following holds.

Theorem 2. *The cardinality of the set of all precomplete classes of deterministic operators is equal to the hypercontinuum.*

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Note: Figure translations are in progress. See original paper for figures.

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