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Let us consider the Hamiltonian equation

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Abstract

Full Text

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On the Stability of Solutions of Hamiltonian Equations in Hilbert Space with Unbounded Periodic Operator Coefficients

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Let us consider the Hamiltonian equation

$$J\frac{dx}{dt} = H(t)x \tag{1}$$

in the complete separable complex Hilbert space $\mathfrak{H}$. Here J is a bounded operator with bounded inverse, anti-Hermitian ($J^* = -J$); $H(t)$ is an unbounded symmetric T-periodic operator, subject to certain general conditions formulated below. Equation (1) often occurs in applications ([1]). In the case of a finite-dimensional space $\mathfrak{H}$, M. G. Krein found sufficient conditions for strong stability of the solutions of equation (1) in terms of the type of points of the spectrum of the monodromy operator ([2]). I. M. Gelfand and V. B. Lidskii ([3]) showed that these conditions are also necessary (for another proof see ([4])). In the present paper these results, with natural modifications, are carried over to equation (1).

1°. We shall assume that $H(t)$ is equal to the sum of two operators

$$H(t) = H_0(t) + H_1(t), \tag{2}$$

where $H_0(t)$ is a self-adjoint positive-definite operator, $H_0^*(t) = H_0(t) \geq \beta I$ ($\beta = \text{const} > 0$), and the following condition is satisfied:

A. The domain of definition $D(H_0^{1/2})$ of the positive square root $H_0^{1/2}(t)$ of the operator $H_0(t)$ is constant, and for $t, t' \in [0, T]$, for any elements $x, y \in D(H_0^{1/2})$, the estimate

$$\left|\left(H_0^{1/2}(t)x, H_0^{1/2}(t)y\right) - \left(H_0^{1/2}(t')x, H_0^{1/2}(t')y\right)\right| \leq$$

$$\leq \text{const} \cdot |t - t'| \, \|H_0^{1/2}(t)x\| \cdot \|H_0^{1/2}(t)y\|$$

holds.

The operator $H_1(t)$ is, in the following sense, subordinated to the operators J and $H_0(t)$:

B. For almost all $t \in [0,T]$, the symmetric operator $H_1(t)$ is defined on the set $D(H_0^{1/2})$; the operator

$$A(t) = H_0^{1/2}(0) J^{-1} H_1(t) H_0^{-1/2}(0), \tag{3}$$

is meaningful, is bounded for almost all $t \in [0,T]$, is strongly measurable on $[0,T]$, and $\|A(t)\| \in \mathcal{L}(0,T)$.

Definition 1. We shall call a function $x(t)$ a **generalized solution** of equation (1) if it has the following properties:

1) $x(t) \in D(H_0^{1/2})$ for all t, and the function $H_0^{1/2}(0)x(t)$ is weakly continuous;

2) for almost all t, the function $H_0^{-1/2}(0)Jx(t)$ is strongly differentiable, and $x(t)$ satisfies, for these t, the equation

$$\frac{d}{dt}\left[H_0^{-1/2}(0)J(x,t)\right] = \left[H_0^{1/2}(t)H_0^{-1/2}(0)\right]^* H_0^{1/2}(t)x(t) + H_0^{-1/2}(0)H_1(t)x(t)$$

and, by weak continuity, the initial condition.

Theorem 1. *If conditions A and B are satisfied for the operator (2), then equation (1) with the initial condition* $x(0) \in D(H_0^{1/2})$ *has a unique generalized solution* $x(t)$. *The resolving operator* $X(t)$, *defined by the formula* $X(t)x(0) = x(t)$, *is representable in the form* $X(t) = H_0^{-1/2}(0)z(t)H_0^{1/2}(0)$.

The operator $Z(t)$ is bounded together with its inverse uniformly in $t \in [0,T]$ and satisfies the relation $Z^*(t)FZ(t) = F$, where

$$F = iH_0^{-1/2}(0)JH_0^{-1/2}(0). \tag{4}$$

By means of the resolution of the identity E_λ of the self-adjoint operator F, define the bounded operator

$$|F|^{1/2} = \int_{-\infty}^{+\infty} |\lambda|^{1/2}\, dE_\lambda, \tag{5}$$

whose inverse is unbounded.

Let the operator $H_0(t)$ in formula (2) be fixed, and let M be the set of all operators $H_1(t)$ satisfying condition B. Introduce in M the distance between the operators $H_1 = H_1(t)$ and $\widetilde{H}_1 = \widetilde{H}_1(t)$ by the formula

$$\rho(H_1, \widetilde{H}_1) = \int_0^T \left\| |F|^{1/2}(A(t) - \widetilde{A}(t))|F|^{-1/2} \right\| dt, \tag{6}$$

where the operator $A(t)$ is defined by formula (3), and

$$\widetilde{A}(t) = H_0^{1/2}(0) J^{-1} \widetilde{H}_1(t) H_0^{-1/2}(0).$$

We note that for any pair of operators $H_1(t)$ and $\widetilde{H}_1(t)$ in M, $\rho(H_1, \widetilde{H}_1) < \infty$. The latter assertion follows from the estimate

$$\left\| |F|^{1/2}(A(t) - \widetilde{A}(t))|F|^{-1/2} \right\| \leq \|A(t) - \widetilde{A}(t)\|,$$

valid for almost all t.

Definition 2. Equation (1) is called **stable** if, for every generalized solution $x(t)$ of it, the estimate

$$\left\| |F|^{1/2} H_0^{1/2}(0) x(t) \right\| \leq \text{const} \cdot \left\| |F|^{1/2} H_0^{1/2}(0) x(0) \right\| \tag{7}$$

holds.

If, in addition, all equations (1) are stable under small changes of the operator $H_1(t)$ in the metric (6), then equation (1) is called **strongly stable**.

The main result is formulated in the following theorem.

Theorem 2. Suppose that on the period $[0, T]$ the coefficients of equation (1) satisfy the conditions of Theorem 1; $X(t)$ and F are the operators defined in Theorem 1. In order that equation (1) be strongly stable, it is sufficient that the spectrum of the operator $Y(T) = |F|^{1/2} Z(T) |F|^{-1/2}$ have no points of mixed kind.

The definition of the kind of points of the spectrum of the operator $Y(T)$ is analogous to the finite-dimensional case and will be given below.

There exists an example showing that if, in inequality (7), the operator $|F|^{1/2}$ is replaced by the identity operator I, then the assertion of Theorem 2 loses its force.

2°. **Definition 3.** If the operator Z is bounded together with its inverse and satisfies the relation $Z^* F Z = F$, then it is called **F-unitary**.

Definition 4. A bounded projector P satisfying the relation $FP = P^* F$ is called a **projector of the first kind** (a **projector of the second kind**) with respect to the operator F, if $(FPx, x) > 0$ ($(FPx, x) < 0$) for every $x = Px \neq 0$.

Definition 5. Suppose the entire spectrum of the operator Z can be surrounded by a finite number of closed contours Γ_j, pairwise nonintersecting and not intersecting the spectrum of Z, each of which is located symmetrically with respect to the unit circle. Suppose that each of the projectors

$$P_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\xi I - Z)^{-1} \, d\xi,$$

is a projector of the first or second kind with respect to F. Then one says that the F-unitary operator Z has no points of spectrum of mixed kind.

Definition 6. The operator Z is called **stable** if

$$\left\| |F|^{1/2} Z^n |F|^{-1/2} x \right\| \leqslant \text{const} \cdot \|x\| \qquad \left(x \in D\left(|F|^{-1/2}\right)\right)$$

for $n = \pm 1, \pm 2, \ldots$. If, in addition, all F-unitary operators from some neighborhood of Z in the uniform operator topology are stable, then Z is called **strongly stable**.

One can give an example of an F-unitary operator Z, having no spectral points of mixed type, for which the set of norms of integral powers is unbounded. However, the following holds:

Theorem 3. *In order that an F-unitary operator Z be strongly stable, it is sufficient that it have no spectral points of mixed type.*

Theorem 4. *If Z is an F-unitary operator, then the operators*

$$Y = |F|^{1/2} Z |F|^{-1/2}, \qquad Y^{-1} = |F|^{1/2} Z^{-1} |F|^{-1/2}, \tag{8}$$

defined on the set $D(|F|)^{-1/2}$, are bounded. If λ and $(\overline{\lambda})^{-1}$ are regular points of the operator Z, then they will also be regular points for the operator Y.

It is now clear that Theorem 3 asserts the boundedness of the powers of the operators Y and Y^{-1}, obtained from the operator Z by formulas (8).

By means of the resolution of the identity E_λ of the operator F, introduce the bounded symmetric operator, together with its inverse,

$$G = \int_0^{+\infty} dE_\lambda - \int_{-\infty}^0 dE_\lambda. \tag{9}$$

From Theorem 4 there follows

Corollary. *If an F-unitary operator Z has no spectral points of mixed type, then the G-unitary operator Y, obtained by formulas (8) from the operator Z, has no spectral points of mixed type.*

Examples show the possibility of a case in which an F-unitary operator Z has spectral points of mixed type, while the G-unitary operator Y, defined from Z by formulas (8), has no spectral points of mixed type. In particular, this indicates that Theorem 3, applied to the G-unitary operator Y, has a wider range of application than the same theorem applied to the F-unitary operator Z. Since the operator G is bounded together with its inverse, Theorem 3, applied to the G-unitary operator Y, asserts the boundedness of the powers of the operator Y.

It can be shown that small changes of the operator $H_1(t)$ in the metric (6) correspond to small changes, in the uniform operator topology, of the G-unitary operator $Y(t)$ defined in Theorem 2. Using Theorem 4 and Theorem 3 applied to the G-unitary operator $Y(t)$, it is easy to prove Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.