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Abstract

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MATHEMATICS

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ON A NUMERICAL METHOD FOR SOLVING LINEAR INTEGRAL EQUATIONS OF CONVOLUTION TYPE

(Presented by Academician M. A. Lavrent' ev, 22 VI 1963)

Suppose it is required to solve the linear Fredholm integral equation of the first kind

$$f(x) = \frac{\lambda}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\xi) K(x-\xi)\, d\xi, \tag{1}$$

where $K(x)$ is an even or odd function of x, for which an analytic representation is known.

Assume that the existence of a solution has already been established and that it is known that all the functions $f(x), \varphi(x)$, and $K(x)$ belong to $L(-\infty, +\infty)$, so that the question concerns only the construction of a numerical method for finding the values of the function $\varphi(x)$ from the given values of $f(x)$, $-\infty \leq x \leq +\infty$. For equations of the first kind, the question of constructing reliable computational schemes is of great importance in connection with the known instability of solutions of such equations (arbitrarily large functions $\varphi(x)$ may correspond to small variations of the function $f(x)$). The degree of instability of the solution can be substantially reduced by an appropriate choice of the computational scheme.

The method for constructing computational schemes set forth below is effective if, for the Fourier transform $F(\omega)$ of the function $f(x)$, we know a sufficiently accurate majorant estimate

$$|F(\omega)| \leq KH(\omega), \qquad -\infty \leq \omega \leq +\infty, \tag{2}$$

where $H(\omega)/Q(\omega) \in L(-\infty, +\infty)$, and $H(\omega)$ is a function given by an analytic expression for all ω.

The essence of the method is as follows: we seek an approximate solution of equation (1) in the form of a linear combination of the values $f(x + x_k)$, $x_k = k\Delta x$, $k = 0, \pm 1, \pm 2, \ldots$:

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$$\varphi(x) \approx \overline{\varphi(x)} = \sum_{-N}^{+N} c_k f(x + x_k); \tag{3}$$

$$c_k = c_{-k}, \qquad \text{if } K(x) = K(-x);$$

$$c_k = -c_{-k}, \qquad \text{if } K(x) = -K(-x).$$

As indicators of the error of the computational formula (1), choose the quantities

$$E_N^{(1)}(\Delta x) = \max_x |\delta\varphi(x)| = \max_{-\infty \le x \le +\infty} |\varphi(x) - \overline{\varphi(x)}|, \tag{4}$$

$$E_N^{(2)}(\Delta x) = \int_{-\infty}^{+\infty} [\delta\varphi(x)]^2 \, dx = \int_{-\infty}^{+\infty} [\varphi(x) - \overline{\varphi(x)}]^2 \, dx. \tag{5}$$

Denote by $F(\omega), \Phi(\omega), Q(\omega), \overline{\Phi(\omega)}$, and $\delta\Phi(\omega)$ the complex Fourier transforms of the functions $f(x), \varphi(x), K(x), \overline{\varphi(x)}$, and $\delta\varphi(x)$, respectively. Obviously, $Q(\omega)$ will be an even and real-valued function or an odd and purely imaginary one, depending on whether $K(x)$ is even or odd.

Since the solution of equation (1) exists and $f(x), \varphi(x)$, and $K(x)$ belong to $L(-\infty, +\infty)$, we have

$$\Phi(\omega) = \frac{F(\omega)}{\lambda Q(\omega)}. \tag{6}$$

From (3), with the aid of the shift theorem for the Fourier transform, we find

$$\overline{\Phi}(\omega) = F(\omega) \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega}, \tag{7}$$

and from the definition of $\delta\varphi(x)$ and (6),

$$\delta\Phi(\omega) = F(\omega) \left\{ \frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right\}. \tag{8}$$

From the last two relations there follow the equalities

$$\delta\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) \left\{ \frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right\} e^{-i\omega x} \, dx, \tag{9}$$

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$$\int_{-\infty}^{+\infty} [\delta\varphi(x)]^2\, dx = \pm \int_{-\infty}^{+\infty} |F(\omega)|^2 \left(\frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right)^2 d\omega, \qquad (10)$$

from which, with the aid of (2), we pass to the inequalities*

$$E_N^{(1)}(\Delta x) \leq \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H(\omega) \left| \frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right| d\omega, \qquad (11)$$

$$E_N^{(2)}(\Delta x) \leq \pm K^2 \int_{-\infty}^{+\infty} H^2(\omega) \left(\frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right)^2 d\omega. \qquad (12)$$

We propose to construct computational formulas of the form (3), determining their coefficients c_k either from the condition

$$\int_{-\infty}^{+\infty} H(\omega) \left| \frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right| d\omega = \min, \qquad (13)$$

or from the condition

$$\pm \int_{-\infty}^{\infty} H^2(\omega) \left(\frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right)^2 d\omega = \min\,. \qquad (14)$$

Finding the coefficients c_k from condition (14) reduces, as is known, to solving a system of linear equations with respect to c_k. Finding c_k from condition (13) is a considerably more difficult problem. Therefore it is reasonable to pass from estimate (13) to a cruder, but also more convenient one. Applying to (11) the Cauchy–Bunyakovsky inequality, we find

$$\left(E_N^{(1)}(\Delta x)\right)^2 \leq \pm \frac{KC_H}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H(\omega) \left(\frac{1}{\lambda Q(\omega)} - \sum_{-N}^{+N} c_k e^{-ik\Delta x\omega} \right)^2 d\omega, \qquad (15)$$

where

$$C_H = \int_{-\infty}^{+\infty} H(\omega)\, d\omega. \qquad (16)$$

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Obviously, sufficiently good computational formulas of the form (3) can be obtained by determining their coefficients c_k, in accordance with estimate (15), from the condition

$$\pm\int_{-\infty}^{+\infty} H(\omega)\left(\frac{1}{\lambda Q(\omega)}-\sum_{-N}^{+N} c_k e^{-ik\Delta x\omega}\right)^2 d\omega = \min\,. \tag{17}$$

* It is assumed here that also $H^2(\omega)/Q^2(\omega) \in L(-\infty,+\infty)$.

We shall illustrate the method set forth with examples of integral equations that have applications in the magnetic and gravitational methods of geophysical prospecting: 1) the problem of analytic continuation of a harmonic function into a horizontal layer, and 2) the problem of computing the second vertical derivative of a harmonic function. The construction of the integral equations is due to the author.

Let $U(x,z)$ be a function harmonic for $z > -H(0 < H < +\infty)$, having singularities on the line $z = -H$, and let, for any $z = \text{const} > -H$, $U(x,z) \in L(-\infty,+\infty)$. Suppose further that $U(x,z) \geq 0$ for $z > -H$.

It can be shown that in this case, for the Fourier transform $F(\omega,0)$ of the function $U(x,0)$, the estimate holds

$$|F(\omega,0)| \leq \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U(x,0)\,dx\right)e^{-|\omega|H} = \frac{\|U(x,0)\|}{\sqrt{2\pi}}e^{-|\omega|H}. \tag{18}$$

The problem of constructing the function $U(x,z)$ $(z < 0)$ in the strip $0 > z > -H$ from the given values $U(x,0)$ is reduced to solving the integral equation

$$U(x,0) = \frac{2}{\pi z^2}\int_{-\infty}^{+\infty} V(\xi,z)\,\frac{\pi(x-\xi)}{\operatorname{sh}\dfrac{\pi(x-\xi)}{z}}\,d\xi, \tag{19}$$

for, as is not difficult to show,

$$U(x,z) = 4V(x,z) - 2U(x,0) - U(x,-z), \tag{20}$$

where $U(x,-z)$ is readily reconstructed from the given values $U(x,0)$ by means of the Poisson integral

$$U(x,-z) = -\frac{z}{\pi}\int_{-\infty}^{+\infty}\frac{U(\xi,0)\,d\xi}{(x-\xi)^2+z^2}. \tag{21}$$

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The problem of finding, from the given values $U(x,0)$, the increments (on the interval h) corresponding to the derivative $\partial^2 U(x,z)/\partial z^2\big|_{z=0}$ is reduced to solving the integral equation

$$U(x,0) = \frac{1}{2h}\int_{-\infty}^{+\infty} v(\xi,h)e^{-|x-\xi|/h}\,d\xi, \tag{22}$$

for it is easy to establish that

$$v(\xi,h) = U(x,0) + h^2\frac{\partial^2 U(x,z)}{\partial z^2}\bigg|_{z=0}. \tag{23}$$

We shall construct computational formulas for solving the equations starting from condition (17).

For equation (19)

$$\lambda Q(\omega) = \frac{1}{\operatorname{ch}^2 \omega z/2}, \tag{24}$$

and therefore condition (17), taking into account the evenness of the kernel of the equation, becomes the following ($c_k^* = c_k,\ k \geq 1,\ c_0^* = c_0/2$):

$$\int_{-\infty}^{+\infty} e^{-|\omega|H}\left(\operatorname{ch}^2\frac{\omega z}{2} - 2\sum_{k=0}^{N} c_k^*\cos k\Delta x\omega\right)^2 d\omega = \min. \tag{25}$$

For equation (22)

$$\lambda Q(\omega) = \frac{1}{1+\omega^2 h^2}, \tag{26}$$

and therefore condition (17), taking into account the evenness of the kernel of the equation, becomes the following:

$$\int_{-\infty}^{+\infty} e^{-|\omega|H}\left(1+\omega^2h^2 - 2\sum_{k=0}^{N} c_k^*\cos k\Delta x\omega\right)^2 d\omega = \min. \tag{27}$$

Table 1 gives the values of the coefficients c_k^* for the cases:

$$\frac{H}{\Delta x} = 2, \qquad -\frac{z}{\Delta x} = 1, \qquad \frac{h}{\Delta x} = 1, \qquad N = 5.$$

Table 1

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k	0	1	2	3	4	5
For equation (19)	1,2723	−1,0787	0,4406	−0,1911	0,0792	−0,0233
For equation (22)	2,2473	−2,1640	0,5817	−0,2345	0,0957	−0,0279

Table 2 gives the results of computations using the formulas constructed. Analytic continuation was carried out to the level $z = -0,5$ for the functions

$$U_1(x,z) = \frac{1+z}{x^2+(1+z)^2}, \qquad U_2(x,z) = \frac{\operatorname{arc\,tg}\dfrac{x+1/2}{1+z} - \operatorname{arc\,tg}\dfrac{x-1/2}{1+z}}{2\operatorname{arc\,tg}1/2}. \quad (28)$$

The increments corresponding to the second derivatives were computed for the same functions for the interval $h = 0,5$. In carrying out the analytic continuation, values of the functions $U_i(x,0)$ and $U_i(x,0,5)$, exact to 4 digits, were used. In Table 2 the computed values are marked with an overbar; δ denotes their errors.

Table 2*

x	0	0,5	1,0	1,5	2,0	2,5	3,0
$U_1(x,0)$	1,0000	0,8000	0,5000	0,3077	0,2000	0,1379	0,1000
$\overline{U}_1(x,-0,5)$	2,0005	1,0004	0,4005	0,2041	0,1176	0,0765	0,0415
δ	0,0005	0,0004	0,0005	0,0041	0,0000	−0,0004	−0,0125
$U_2(x,0)$	1,0000	0,8469	0,5599	0,3470	0,2238	0,1530	0,110
$\overline{U}_2(x,-0,5)$	1,7324	1,1941	0,4920	0,2397	0,1326	0,0840	0,0494
δ	0,0384	0,0001	−0,0080	0,0039	−0,0015	−0,0021	−0,0104
$(0,5)^2 \left.\dfrac{\partial^2 \overline{U}_1(x,z)}{\partial z^2}\right\|_{z=0}$	0,5001	0,0641	−0,1250	−0,0837	−0,0440	−0,0233	−0,0167
δ	0,0001	0,0301	0,0000	0,0001	0,0000	0,0032	−0,0037
$(0,5)^2 \left.\dfrac{\partial^2 \overline{U}_2(x,z)}{\partial z^2}\right\|_{z=0}$	0,3521	0,1331	−0,0975	−0,0905	−0,0514	−0,0276	−0,0180
δ	+0,0070	−0,0017	−0,0015	+0,0012	−0,0005	−0,0003	−0,0031

x	3,5	4,0	4,5	5,0	5,5	6,0	6,5
$U_1(x,0)$	0,0755	0,0588	0,0471	0,0385	0,0320	0,0270	0,0231
$\overline{U}_1(x,-0,5)^*$	0,0312	0,0250	0,0215	0,0180	0,0150	0,0128	0,0109
δ	−0,0088	0,0006	0,0017	0,0016	−0,0014	−0,0010	−0,0009
$U_2(x,0)$	0,0828	0,0643	0,0513	0,0419	0,0348	0,0293	0,0251
$\overline{U}_2(x,-0,5)^*$	0,0341	0,0277	0,0228	0,0196	0,0167	0,0137	0,0121
δ	−0,0098	−0,0060	−0,0038	−0,0020	−0,0011	−0,0013	−0,0037
$(0,5)^2 \frac{\partial^2 \overline{U}_1(x,z)}{\partial z^2}\Big\vert_{z=0}$	0,0103	−0,0066	−0,0039	−0,0127	−0,0019	−0,0014	−0,0010
δ	−0,0026	−0,0021	−0,0031	−0,0021	−0,0015	−0,0011	−0,0008
$(0,5)^2 \frac{\partial^2 \overline{U}_2(x,z)}{\partial z^2}\Big\vert_{z=0}$	0,0116	−0,0072	−0,0047	−0,0029	−0,0020	−0,0017	−0,0010
δ	−0,0028	−0,0007	−0,0015	−0,0007	−0,0005	−0,0006	−0,0002

* In the computations, values of $U_i(x,0)$ for $x = 7,0\ (0,5)\ 9,5$ were also used.

The results of the computations indicate sufficient accuracy of the constructed computational formulas, although they are undoubtedly not the best ones (condition (17), considerably cruder in comparison with (13), was used, and the number of points employed, N, was relatively small, $N = 5$).

In conclusion, we note that the method for constructing computational formulas is readily generalized to the case of integral equations of convolution type for any number of variables.

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Note: Figure translations are in progress. See original paper for figures.

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