Reports of the Academy of Sciences of the USSR

1963

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1963. Volume 148, No. 3

MATHEMATICS

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ON QUASIPROJECTIVE CLASSES OF MODELS

(Presented by Academician A. I. Mal' tsev on 30 VII 1962)

We shall consider models of an arbitrary fixed type, in which the first basic relation is relativized equality.

Following ([1,2]), by the **reduced product** of a family of models $(\mathfrak{M}_i)_{i\in I}$ with respect to a filter F on the set I we shall mean the model $\Pi\mathfrak{M}_i/F$, whose underlying set is the Cartesian product of the underlying sets $\mathfrak{M}_i$, and whose basic relations R_ρ are such that

$$\langle f_1, \dots, f_{n_\rho}\rangle \in R_\rho \leftrightarrow \{i : \langle f_1(i), \dots, f_{n_\rho}(i)\rangle \in R_\rho^{(i)}\} \in F.$$

If F is an ultrafilter, then $\Pi\mathfrak{M}_i/F$ is called the **ultraproduct** of the family $(\mathfrak{M}_i)_{i\in I}$. Abstract classes of models closed with respect to ultraproducts will be called **quasiprojective**.

It is known that axiomatizable classes are quasiprojective ([1]). Hence follows the quasiprojectivity of all projective, in the sense of ([3]), classes.

The reduced product of a family $(\mathfrak{M}_i)_{i\in I}$ with respect to a filter F is determined, up to isomorphism, by any subfamily $(\mathfrak{M}_i)_{i\in J}$ such that $J \in F$. Hence follows

Theorem 1. *Let K be a quasiprojective class. Then for any family of models $(\mathfrak{M}_i)_{i\in I}$ and any ultrafilter F on I, from $\{i : \mathfrak{M}_i \in K\} \in F$ it follows that $\Pi\mathfrak{M}_i/F \in K$.*

It follows from Theorem 1 that the union of a finite number and the intersection of any number of quasiprojective classes are quasiprojective classes.

Theorem 2. *Let a family of quasiprojective classes be such that the intersection of any finite subfamily of it is nonempty. Then the intersection of the whole family is nonempty.*

From the last remark it follows that Theorem 2 is equivalent to the following proposition:

The intersection of every chain of nonempty quasiprojective classes is nonempty.

Let us prove this proposition. Let $(K_i)_{i \in I}$ be a chain of nonempty quasiprojective classes, and let I be ordered so that $i \leq j \leftrightarrow K_j \subset K_i$. In each class K_i choose arbitrarily a model $\mathfrak{M}_i$. Let F be an ultrafilter on I containing all possible sets $\{j : i \leq j\}$ for all $i \in I$. For every class K_j ($j \in I$) we have $\{i : \mathfrak{M}_i \in K_j\} \in F$. Then, by Theorem 1, $\Pi \mathfrak{M}_i / F \in K_j$, whence

$$\bigcap_{j \in I} K_j \neq \emptyset.$$

From Theorem 2 and from the quasiprojectivity of axiomatizable classes follows the Gödel–Mal' tsev theorem.

The homomorphic, strongly homomorphic, and multiplicative closures of a quasiprojective class are quasiprojective. Also the following holds.

Theorem 3. *If a class K is quasiprojective, then $S(K)$ is a universally axiomatizable class.*

With the help of results of A. D. Taimanov ([4]), this theorem can be strengthened. From Theorem 2 it follows that a quasiprojective class of finite type is an l-class for any l. Hence it follows that the class $S_{(l)}(K)$ of all l-submodels from K belongs to

$$\underbrace{UE \dots C_{\Delta}}_{l}.$$

We generalize the concept of an l-submodel to the case of models of arbitrary types. By $\mathfrak{M}^{(R)}$, where R is some set of basic predicates, we shall denote the R-reduction of the model $\mathfrak{M}$. Let the models $\mathfrak{M}$ and $\mathfrak{N}$ be such that, for any finite set R of basic predicates, $\mathfrak{N}^{(R)} \leqslant [l] \mathfrak{M}^{(R)}$ (in the sense of (4)), and $\mathfrak{N}$ is a submodel of $\mathfrak{M}$. Then we shall call $\mathfrak{N}$ an l-submodel of the model $\mathfrak{M}$.

Theorem 4. *Let K be a quasiprojective class. Then, for every l,*

$$S_{(l)}(K) \in UE \dots C_{\Delta}.$$

Obviously, for every cardinality $\aleph$, every infinite model of a quasiprojective class K is embeddable in a model from K of cardinality greater than $\aleph$. With the aid of the generalized continuum hypothesis one proves the stronger

Theorem 5. *Every infinite model $\mathfrak{M}$ of a quasiprojective class K has in it an extension of any cardinality greater than the cardinality of $\mathfrak{M}$.*

Theorem 5 strengthens some known results on axiomatizable and projective classes ([3,5]).

From Theorem 6.5 of (4) it follows that, if a quasiprojective class contains finite models with arbitrarily large numbers of elements, then it contains continuum models (cf. (5)).

For every nondenumerable (infinite) cardinality $\aleph$ one has $\aleph^{\aleph_0} = \aleph$. Hence it follows:

Theorem 6. *An infinite model of nondenumerable cardinality of a quasiprojective class has in it a proper extension of the same cardinality.*

This theorem gives a partial answer to a question posed by A. I. Mal' tsev (6).

A class of models K will be called **locally finite** if, for every model $\mathfrak{M}$ from K, every finite submodel of it is embeddable in a finite K-submodel of $\mathfrak{M}$, and **uniformly locally finite** if, for every natural number n, there exists a natural $f(n)$ such that, for every model $\mathfrak{M}$ from K, every n-element submodel of it is embeddable in an $f(n)$-element submodel of the model $\mathfrak{M}$. Obviously, if a class is uniformly locally finite, then it is locally finite.

Theorem 7. *If a quasiprojective class of finite type is locally finite, then it is uniformly locally finite.*

The validity of this theorem is easily proved with the aid of the following theorem:

Theorem 8. *If a quasiprojective class K of finite type, for every natural number n, contains a minimal model with a number of elements greater than n, then it contains an infinite model having no finite K-submodels.*

By modifying the operation of reduced product, one can obtain an elegant characterization of universally axiomatizable classes. Let M be a nonempty set; $(\mathfrak{M}_i)_{i\in I}$ a family of models defined on M; F a filter in I. By the reduced subproduct of the family $(\mathfrak{M}_i)_{i\in I}$ with respect to F we shall mean the model defined on M, $\Pi^*\mathfrak{M}_i/F$, whose basic relations R_ρ are such that

$$\langle a_1, \dots, a_{n_\rho}\rangle \in R_\rho \leftrightarrow \{\, i : \langle a_1, \dots, a_{n_\rho}\rangle \in R_\rho^{(i)} \,\} \in F.$$

If F is an ultrafilter, then $\Pi^*\mathfrak{M}_i/F$ will be called the ultrasubproduct of the family $(\mathfrak{M}_i)_{i\in I}$. With the aid of results from (7) it is easily proved that

Theorem 9. *A class of models is universally axiomatizable if and only if it is closed under ultrasubproducts.*

With the aid of this theorem, many results connected with universally axiomatizable classes can be formulated in the language of reduced subproducts. For example, an obvious necessary and sufficient condition for universal axiomatizability of multipli-

···of the relative closure of a class of models is its closure with respect to all possible reduced subdirect products. Let us give this condition a weaker form.

A filter in an infinite set will be called a generalized Fréchet filter if it consists of all possible subsets whose complements have smaller cardinalities.

Theorem 10. *A class of models is universally axiomatizable and multiplicatively closed if and only if it is closed with respect to reduced subproducts relative to generalized Fréchet filters.*

We note that classes of models closed with respect to reduced products relative to generalized Fréchet filters are, in their properties, related to quasiprojective classes. For such classes, for example, Theorems 1, 2, 3, 5, 6 are valid.

Applications.

1. A. I. Mal' cev showed that the classes of RN-, RI-, and Z-groups are axiomatizable ([3]). This result is generalized by Theorem 11 below, whose proof is based on the use of properties of model correspondences invariant with respect to ultraproducts.

Let $\mathfrak{R}$ be some class of groups. We shall call a group an $\mathfrak{R}N$-group if it has a normal system whose factors belong to $\mathfrak{R}$. The classes of $\mathfrak{R}I$- and $\mathfrak{R}Z$-groups are defined analogously.

Theorem 11. *If $\mathfrak{R}$ is a quasiprojective class of groups, then the classes of $\mathfrak{R}N$-, $\mathfrak{R}I$-, and $\mathfrak{R}Z$-groups are quasiprojective.*

From Theorems 3 and 11 it follows that if $\mathfrak{R}$ is a universally axiomatizable class, then the same is true for the classes of $\mathfrak{R}N$-, $\mathfrak{R}I$-, and $\mathfrak{R}Z$-groups.

2. Let K be some class of models, and let L be the largest of those classes M for which $H(M) \subset K$. Then, if the complement of K is closed with respect to ultrapowers, the complement of L is also closed with respect to ultrapowers.

Since the class of $\overline{RI}$-groups is the largest of the classes of groups M such that $H(M) \subset RI \in AC_{\Delta}$, its complement is closed with respect to ultrapowers. The same is true for the class of $\overline{Z}$-groups. Hence it follows that

The axiomatizability of the classes of $\overline{RI}$- and $\overline{Z}$-groups is equivalent to their quasiprojectivity.

The class of $\overline{RI}$-groups is homomorphically closed. It is also closed with respect to direct sums. Consequently, for its axiomatizability, closure with respect to direct products is necessary (this is also true for the class of $\overline{RN}$-groups). The latter is also sufficient, since if a class of models is closed with respect to homomorphisms and direct products, then it is quasiprojective. Thus:

The class of $\overline{RI}$-groups is axiomatizable if and only if it is closed with respect to direct products.

In the case of a positive solution to Question XX from ([8]), the same is true for the class of $\overline{Z}$-groups.

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Received
30 VI 1962

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Note: Figure translations are in progress. See original paper for figures.