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## Abstract

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# THE FUNCTIONAL SPACE $K_\Lambda^m(\Omega)$.

# SOME APPLICATIONS TO THE THEORY OF BOUNDARY-VALUE PROBLEMS

*(Presented by Academician S. L. Sobolev on 22 II 1963)*

In the present article we introduce and study the functional space $K_\Lambda^m(\Omega)$, which is a subspace of the known spaces of S. L. Sobolev $W_p^l$ (1) and $W^m_{p,\alpha_1,\dots,\alpha_s}$ (2,3). The properties of the functions of the latter spaces, established by S. L. Sobolev (1,4,5), V. I. Kondrashov (2,6), L. D. Kudryavtsev (3), and subsequently by other authors, as well as their additional properties obtained in the present work, play an essential role throughout this investigation. In addition, the theory of abstract Hilbert space is used to a considerable extent.

In $K_\Lambda^m(\Omega)$, by means of specially introduced functionals, a certain set of completely continuous operators $G_{\beta,\gamma}$ is defined. The results of the study of these operators are applied to the solution of eigenvalue problems for a certain class of elliptic equations under very diverse boundary conditions, at least in a generalized sense.

The method of investigating eigenvalue problems based on the theory of abstract Hilbert space has been used before. As an example one may cite the work of Sandgren (7), in which eigenvalue problems for the Laplace equation are studied. In the present work this method is developed and applied to the theory of boundary-value problems for elliptic equations of various orders,* as well as for equations degenerating on the boundary of the domain in which they are considered, with a variety of boundary conditions.

1. Let the functions $u(x_1,\dots,x_n)$ have inside $\Omega$ all generalized (in the sense of S. L. Sobolev) partial derivatives up to order $m$ inclusive; moreover, both the functions themselves and all their partial derivatives up to order $m-1$ inclusive are summable in $\Omega$ to the power $p$, and

$$\int_\Omega \prod_{l=1}^{s} \sigma_l \left( \sum_{\sum m_i = m} \left| \frac{\partial^m u}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \right| \right)^p d\Omega < \infty. \qquad (1)$$

Here $\Omega$ is a bounded domain of the $n$-dimensional Euclidean space $E^{(n)}$, which is a Lipschitz image of a cube, with sufficiently smooth $(n-1)$-dimensional boundary $\Gamma$**; $\sigma_l$ are positive, sufficiently differentiable in $\Omega$ functions having power order $\alpha_l$ of decrease when approaching the corresponding piece of the boundary $\Gamma_l$, $\alpha_l < p-1$.

Following V. I. Kondrashov (2), we shall say that the functions $u(x_1, \dots, x_n)$ defined above belong to the space $W^m_{p,\alpha_1,\dots,\alpha_s}(\Omega)$.

**Lemma 1**. Suppose that on the boundary $\Gamma$ of the domain $\Omega$ a system of functions*

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* Some of the problems considered here were also studied earlier (2,14) by the variational method.
** $\Gamma$ may also be piecewise smooth.
*** For $m = 1$, $p = 2$, $\tau$ essentially bounded, and with the volume integral without weight, the corresponding inequality was obtained by Sandgren (7). The present inequality is proved by another method, different from that used in (7).

$$\tau_{\gamma_1,\dots,\gamma_n}(x), \text{ for which } \int_\Gamma \tau_{\gamma_1,\dots,\gamma_n}\, d\Gamma \neq 0 \quad \text{and}$$

$$\tau_{\gamma_1,\dots,\gamma_n}(x) \in L_q(\Gamma), \qquad \text{if } \sum_i \gamma_i < m-1,$$

$$\tau_{\gamma_1,\dots,\gamma_n}(x) \in L_\chi(\Gamma), \qquad \text{if } \sum_i \gamma_i = m-1,$$

where $\alpha_i/p < 1/q - 1/\chi$ $(i = 1, 2, \dots, s)$, $1/p + 1/q = 1$.

Then for all $u(x_1, \dots, x_n) \in W^m_{p,\alpha_1,\dots,\alpha_s}(\Omega)$ one has

$$\int_\Omega |u|^p\, d\Omega \le C \left[ \sum_{\gamma=0}^{m-1} \sum_{\sum_i \gamma_i = \gamma} \left| \int_\Gamma \tau_{\gamma_1,\dots,\gamma_n} \frac{\partial^\gamma u}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}\, d\Gamma \right|^p + \right.$$

$$\left. + \int_\Omega \prod_{l=1}^s \sigma_l \left( \sum_{\sum_i m_i = m} \left| \frac{\partial^m u}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \right|^p \right) d\Omega \right], \qquad C = C(\tau_{\gamma_1,\dots,\gamma_n}). \tag{2}$$

The lemma is proved by obtaining a chain of auxiliary inequalities and applying a method developed by S. M. Nikol' skii (8) (see also (9)).

**Remark.** On the basis of the corresponding embedding theorems for the spaces $W^l_p$, the conditions of the lemma may be sharpened by imposing weaker requirements on the exponent $q$ (1, 6).

**2.** Let now $u(x_1, \dots, x_n) \in W^m_{2,\alpha_1,\dots,\alpha_s}$ in a domain $\Omega$ with boundary $\Gamma$. Consider on the boundary $\Gamma$ of the domain $\Omega$ the system of functions

$$\Lambda = \{\theta_{\beta_1,\dots,\beta_n};\ \tau_{\gamma_1,\dots,\gamma_n}\}. \tag{3}$$

Here it is allowed that part of the functions of the system be prescribed on pieces (possibly overlapping) of the boundary $\Gamma$, while another part is prescribed on the whole boundary $\Gamma$. In addition, the requirements on the functions from the system $\Lambda$ correspond to the conditions of Lemma 1 with the corresponding remark.

That function of the system $\Lambda$ which is prescribed on the part $\Gamma_{\beta_1,\dots,\beta_n}$ of the boundary $\Gamma$ is denoted by the symbol $\theta_{\beta_1,\dots,\beta_n}$, and to it are put in correspondence derivatives

$$\frac{\partial^\beta u}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}; \qquad \sum_i \beta_i = \beta; \quad \beta \le m-1,$$

of such functions for which these derivatives on the remaining part of the boundary $\Gamma - \Gamma_{\beta_1,\dots,\beta_n}$ vanish in the mean with certain exponents corresponding to the embedding theorems for the spaces under consideration, and, in addition,

$$\int_{\Gamma_{\beta_1,\dots,\beta_n}} |\theta_{\beta_1,\dots,\beta_n}|\, d\Gamma \neq 0.$$

The function of the system $\Lambda$ which is prescribed on the whole boundary $\Gamma$ is denoted by the symbol $\tau_{\gamma_1,\dots,\gamma_n}$. To it are put in correspondence the derivatives $\partial^\gamma u/\partial x_1^{\gamma_1}\dots\partial x_n^{\gamma_n}$ of such functions $u \in W^m_{2,\alpha_1,\dots,\alpha_s}(\Omega)$ which satisfy the condition

$$\int_\Gamma \tau_{\gamma_1,\dots,\gamma_n} \frac{\partial^\gamma u}{\partial x_1^{\gamma_1}\dots\partial x_n^{\gamma_n}}\, d\Gamma = 0, \qquad \sum_i \gamma_i = \gamma, \quad \gamma \le m-1.$$

Define the linear functional space as follows:

$$K^m_\Lambda(\Omega) = \left\{ u \in W^m_{2,\alpha_1,\dots,\alpha_s}(\Omega);\ \left.\frac{\partial^\beta u}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}\right|_{\Gamma-\Gamma_{\beta_1,\dots,\beta_n}} = 0; \right.$$

$$\left. \int_\Gamma \tau_{\gamma_1,\dots,\gamma_n} \frac{\partial^\gamma u}{\partial x_1^{\gamma_1}\dots\partial x_n^{\gamma_n}}\, d\Gamma = 0 \right\}$$

($\beta, \gamma \le m-1$ and for at least one $k$, $\beta_k$ and $\gamma_k$ are pairwise unequal; the functions $\tau_{\gamma_1,\dots,\gamma_n}$, in particular, may be constants). In $K^m_\Lambda(\Omega)$ one defines a sca-

scalar product

$$((u,v))_{m,\alpha_1,\dots,\alpha_s} = \int_\Omega \prod_{l=1}^{s} \sigma_l \left( \sum_{\Sigma m_i = m} C_{m_1,\dots,m_n}(x) \frac{\partial^m u}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \frac{\partial^m v}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \right) d\Omega \tag{4}$$

for all $u, v \in K_\Lambda^m(\Omega)$.

$C_{m_1,\dots,m_n}(x) \geqslant 0$ are bounded summable functions such that

$$\int_\Omega \prod_{l=1}^{s} \sigma_l \left[ \sum_{\Sigma m_i = m} C_{m_1,\dots,m_n}(x) \left( \frac{\partial^m u}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \right)^2 \right] d\Omega \geqslant$$

$$\geqslant C \int_\Omega \prod_{l=1}^{s} \sigma_l \cdot \sum_{\Sigma m_i = m} \left( \frac{\partial^m u}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \right)^2 d\Omega, \qquad C > 0 \text{ (constant)}.$$

The form (4) has been chosen for simplicity of exposition. The scalar product in $K_\Lambda^m(\Omega)$ can also be introduced in various other ways; we shall not dwell on this here. With the aid of (4), a norm is introduced in $K_\Lambda^m(\Omega)$.

**Theorem 1.** *The space $K_\Lambda^m(\Omega)$ is complete.*

The proof follows the schemes developed in papers ($^{2-4}$), with the use of Lemma 1. $K_\Lambda^m(\Omega)$ is separable. Thus the space $K_\Lambda^m(\Omega)$ is a complete separable Hilbert space.

3. In $K_\Lambda^m(\Omega)$ a system of bilinear functionals is defined:

$$[u,v]_{\beta,\gamma} = \sum_{\beta}{}' \sum_{\Sigma\beta_i = \beta}{}' \int_{\Gamma_{\beta_1,\dots,\beta_n}} \theta_{\beta_1,\dots,\beta_n} \left( \frac{\partial^\beta u}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \frac{\partial^\beta v}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right) d\Gamma +$$

$$+ \sum_{\gamma}{}' \sum_{\Sigma\gamma_i = \gamma}{}' \int_{\Gamma} \tau_{\gamma_1,\dots,\gamma_n} \left( \frac{\partial^\gamma u}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}} \frac{\partial^\gamma v}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}} \right) d\Gamma, \tag{5}$$

where the primes on the sums indicate the possible absence of some terms, and

$$(u,v)_{\beta,\gamma} = ((u,v))_{m,\alpha_1,\dots,\alpha_s} + [u,v]_{\beta,\gamma} \tag{6}$$

for all $u, v \in K_\Lambda^m(\Omega)$. The meaning of the notation in (6) is clear from (4) and (5).

**Lemma 2.** *In the case of nonnegative $\theta_{\beta_1,\dots,\beta_n}$ and $\tau_{\gamma_1,\dots,\gamma_n}$, the norms*

$$|u|_{\beta,\gamma} = [(u,u)_{\beta,\gamma}]^{1/2}$$

*and*

$$\|u\|_{\beta,\gamma} = [((u,u))_{m,\alpha_1,\dots,\alpha_s}]^{1/2}$$

*are equivalent in* $K_\Lambda^m(\Omega)$.

The proof is carried out by the method used in paper ($^7$), with the use of results from papers ($^{2,4,6,10}$), as well as Lemma 1.

**Theorem 2.** *The symmetric bilinear functional* $[u,v]_{\beta,\gamma}$ *is bounded in* $K_\Lambda^m(\Omega)$, *i.e., there exists a constant* $C$ *(independent of* $u, v$*) such that*

$$|[u,v]_{\beta,\gamma}| \leqslant C\|u\|_{\beta,\gamma}\|v\|_{\beta,\gamma}.$$

The theorem is proved with the aid of Hölder' s inequality and Lemma 2.

4. From Theorem 2, on the basis of the corresponding theorem of functional analysis ($^{11}$), it follows that the functionals (5) define in $K_\Lambda^m(\Omega)$ corresponding bounded linear operators $G_{\beta,\gamma}$ by means of the equality

$$[u,v]_{\beta,\gamma} = ((G_{\beta,\gamma}u,v))_{m,\alpha_1,\dots,\alpha_s}, \qquad u,v \in K_\Lambda^m(\Omega). \tag{7}$$

The operators $G_{\beta,\gamma}$ map $K_\Lambda^m(\Omega)$ into $K_\Lambda^m(\Omega)$.

From the symmetry of $[u,v]_{\beta,\gamma}$ follows the self-adjointness of the operators $G_{\beta,\gamma}$.

**Theorem 3.** *The operators* $G_{\beta,\gamma}$ *are completely continuous.*

The proof follows the scheme given in ($^{12}$), with the use of Lemma 1 and the compactness theorems ($^{2,4,6,10}$).

From Hilbert' s theorem ($^{13}$) it follows that each of the operators $G_{\beta,\gamma}$ defined above has a corresponding spectrum of real eigenvalues, as well as an orthogonal basis consisting of eigenfunctions of the corresponding operators.

5. Assuming sufficient smoothness of the coefficients $C_{m_1,\dots,m_n}(x)$, by known methods ($^{5,6}$) one can show that the eigenfunctions of the operators $G_{\beta,\gamma}$ have derivatives in $\Omega$ up to order $2m$ inclusive and satisfy there the equation

$$L^m u \equiv \sum_{\sum_i m_i = m} \frac{\partial^m}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}} \left( \prod_{l=1}^{s} \sigma_l C_{m_1,\dots,m_n}(x) \frac{\partial^m u}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}} \right) = 0\,*,$$

as well as the generalized boundary conditions:

a) on those pieces $\Gamma - \Gamma_{\beta_1,\dots,\beta_n}$, where the derivatives $\partial^\beta u / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}$ vanish, the boundary conditions are assumed in the mean with certain exponents $b$, corresponding to the embedding theorems for the spaces under consideration:

$$\lim_{\Gamma_\eta \to \Gamma - \Gamma_{\beta_1,\dots,\beta_n}} \int_{\Gamma - \Gamma_{\beta_1,\dots,\beta_n}} \left| \frac{\partial^\beta u}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right|^b d\Gamma = 0 \qquad \left( \sum_i \beta_i = \beta \right);$$

b) on the free pieces of the boundary **

$$\lim_{\Gamma_\eta \to \Gamma} \int_{\Gamma_\eta} \Phi \left( \tau_{\gamma_1,\dots,\gamma_n};\, u, v, \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}, \dots, \frac{\partial^{2m-1} u}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \frac{\partial^{m-1} v}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} \right) d\Gamma = 0.$$

If the eigenfunctions are sufficiently differentiable in the classical sense up to the boundary, then, in particular, for the degenerating Laplace equation they are a solution of the problem

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sigma^\alpha \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{in } \Omega,$$

$$\sigma^\alpha \frac{\partial u}{\partial n} + \frac{1}{\mu} \partial u \Big|_{\Gamma - \Gamma_\sigma} = 0, \qquad u\Big|_{\Gamma_\sigma} = 0,$$

where $\Gamma_\sigma$ is the degeneracy portion ($\sigma^\alpha|_{\Gamma_\sigma} = 0$), and $\mu$ is an eigenvalue.

We shall not dwell in detail on the boundary conditions ***.

I take this opportunity to express my gratitude to V. I. Kondrashov for suggesting the topic and for valuable advice.

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## CITED LITERATURE

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
2. V. I. Kondrashov, DAN, **142**, No. 6, 1243 (1962).
3. L. D. Kudryavtsev, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **55** (1959).

4. S. L. Sobolev, Matem. sborn., **4** (46), 3, 471 (1938).

5. S. L. Sobolev, Matem. sborn., **2** (44), 3, 465 (1937).

6. V. I. Kondrashov, DAN, **72**, No. 6, 1009 (1950).

7. L. Sandgren, *Meddelanden från Lunds universitets matematiska seminarium*, **13** (1955).

8. S. M. Nikol' skii, Matem. sborn., **33** (75), 2, 261 (1953).

9. A. A. Vasharin, DAN, **117**, No. 5, 742 (1957).

10. I. A. Solomeshch, Matem. sborn., **54** (96), 3, 295 (1961).

11. V. I. Smirnov, *A Course of Higher Mathematics*, 5, 1950.

12. F. Riesz, B. Sz.-Nagy, *Lectures on Functional Analysis*, IL, 1954.

13. G. E. Shilov, *Mathematical Analysis*, 2nd ed., Moscow, 1960.

14. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 2, Moscow-Leningrad, 1951.

15. V. I. Kondrashov, DAN, **90**, No. 2, 129 (1953).

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* The requirements on the operator $L^m$ correspond to the requirements imposed on the corresponding operators in works ($^{2,15}$).

** For any $v \in K_\Lambda^m(\Omega)$. Here $\sum_i m_i = 2m - 1,\ \sum_i \mu_i = m - 1$.

*** In some cases the results of the present work can be obtained in a Riemannian space and an estimate can be given for the growth of the eigenvalues.

*Note: Figure translations are in progress. See original paper for figures.*

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