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Abstract

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RECURRENCE, PERIODICITY, AND TRANSITIVITY IN DYNAMICAL SYSTEMS WITHOUT UNIQUENESS

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In this paper we consider certain properties of recurrence, periodicity, stability in the sense of Poisson, and transitivity in semigroups of multivalued mappings (s.m.m.) (R, S, f) ([1]), where R is a uniform separated space ([2]).

1°. **Definition 1** ([3]). A point $p_0 \in R$ will be called a **point of discontinuity** for the s.m.m. (R, S, f) if, for any $s \in S$, the mapping f^s of the space R into itself, defined by the condition

$$f^s(p) = f(p, s) \quad (p \in R),$$

is strongly discontinuous at the point $p = p_0$ ([4]).

Definition 2. A set $A \subseteq R$ will be called **transitive** ([5]) if

$$A \subseteq \overline{f(p, S)}$$

for every point $p \in A$.

A closed semi-invariant ([1]) transitive set is minimal semi-invariant. If all points of a minimal semi-invariant set Σ are points of discontinuity, then Σ is a transitive set.

Let Σ be an arbitrary minimal semi-invariant set and $p \in \Sigma$. If the set $F_1 = \overline{f(p, S)} \subseteq \Sigma$ is semi-invariant, then $\Sigma = F_1$; otherwise consider the set

$$F_2 = \overline{f(F_1, S)} \subseteq \Sigma.$$

Suppose that closed sets

$$F_1 \subseteq F_2 \subseteq \dots \subseteq F_\alpha \subseteq \dots \subseteq \Sigma$$

have been constructed for all transfinite numbers α less than some transfinite β. If β is a limiting ordinal number, then define

$$F_\beta = \bigcup_{\alpha < \beta} F_\alpha.$$

If $\beta = \delta + 1$, then define

$$F_\beta = \overline{f(F_\delta, S)}.$$

In both cases, from $\alpha < \beta$ it follows that $F_\alpha \subseteq F_\beta$. Since the set of all closed subsets of Σ can be well ordered, there will be found some transfinite number γ such that

$$F_{\gamma+1} = F_\gamma,$$

i.e.

$$\overline{f(F_\gamma, S)} = F_\gamma;$$

then F_γ is a closed nonempty semi-invariant subset of Σ. Therefore $F_\gamma = \Sigma$.

Thus, to every minimal semi-invariant set Σ and point $p \in \Sigma$ there corresponds some transfinite $\gamma(\Sigma, p)$. Let

$$\gamma(\Sigma) = \min_{p \in \Sigma} \gamma(\Sigma, p).$$

Examples show that, for any countable transfinite α, there is a minimal semi-invariant bicompact set Σ such that

$$\gamma(\Sigma) \geq \alpha.$$

2°. Introduce the notation:

$$T_p = f(p, S) \quad (p \in R);$$

D is the group of real numbers; D^+ is the semigroup of nonnegative numbers; U is the filter of neighborhoods of the uniform structure of the space R ([2]).

Definition 3. We shall say that a point $p \in R$ **satisfies condition** R_i ($i = 1, 2, \ldots, 5$), if for every $\alpha \in U$ there exists a bicompact set $K \subseteq S$ such that condition K_i ($i = 1, 2, \ldots, 5$) is fulfilled:

K_1.

$$T_p \subseteq \alpha[f(p, sK)]$$

for every $s \in S$.

K_2.

$$T_p \subseteq \alpha[f(q, K)]$$

for every point $q \in T_p$.

K_3. For arbitrary elements s_1 and s_2 of S, there exists an element $s_3 \in s_2 K$ such that

$$f(p, s_1) \subseteq \alpha[f(p, s_3)].$$

K_4. For arbitrary elements s_1 and s_2 of S, there exists an element $s_3 \in s_2 K$ such that

$$f(p, s_1) \subseteq \alpha[f(p, s_3)]$$

and

$$f(p, s_3) \subseteq \alpha[f(p, s_1)].$$

K_5. For any element $s \in S$ and any point $q \in T_p$ there exists an element $s' \in K$ such that $f(p, s) \subseteq a[f(q, s')]$.

Each of the properties R_i $(i = 1, 2, \dots, 5)$ of a point $p \in R$ in the case of an ordinary dynamical system ([6]) ($S = D,\ R$ a metric space) is equivalent to the recurrence of the motion issuing from the point p.

Properties R_3 and R_4 (in the case where the space R is metric and $S = D^+$) were introduced by B. M. Budak ([7]) under the names (+)-recurrence and strong (+)-recurrence.

Theorem 1.

$$\begin{array}{ccccc} R_4 & \to & R_3 & \to & R_1 \\ & \uparrow & & \downarrow & \\ & R_5 & \to & R_2 & \end{array}$$

(an arrow denotes logical implication).

Examples show that the conditions R_4 (R_3, R_5, R_5) are stronger than the corresponding conditions R_3 (R_1, R_3, R_2), while the conditions R_2 and R_3 are independent. It is unknown whether condition R_2 (R_5) follows from condition R_4.

Theorem 2. *If Σ is a bicompact minimal semi-invariant set, all of whose points are points of continuity, then any point $p \in \Sigma$ satisfies condition R_2.*

There exist bicompact invariant transitive sets no point of which satisfies condition R_2.

Theorem 3. *If a point p satisfies condition R_2 and the space R is complete, then $\overline{T}_p$ is a bicompact transitive set.*

Examples show that in Theorem 3 condition R_2 cannot be replaced by condition R_3, even if all points of the space are points of discontinuity.

From Theorems 2 and 3 there follows a corollary which is a generalization of well-known theorems of Birkhoff ([6]):

Corollary. *Let R be a complete space all of whose points are points of discontinuity. In order that the set $\Sigma \subseteq R$ be a bicompact minimal semi-invariant set, it is necessary and sufficient that all points $p \in \Sigma$ satisfy condition R_2.*

In the proof of the following theorem, essential use is made of the fact that the set of all bicompact subsets of a bicompact set, endowed with the finite topology, is bicompact ([8]).

Theorem 4. *In complete spaces all of whose points are points of discontinuity, the conditions R_2 and R_5 (R_1 and R_3) are equivalent. Under the same assumptions, condition R_4 is stronger than R_2, and R_2 is stronger than R_3.*

Thus Theorems 2 and 3 are a generalization of the corresponding results of B. M. Budak (7) and M. I. Minkevich (9).

An example of E. A. Barbashin (3) shows that there exist bicompact minimal semi-invariant sets all of whose points are points of discontinuity and no point of which satisfies condition R_4.

3°. Let (R, S, f) be an s.t.s. and $p \in R$. Introduce the notation:

$$S_1(p) = \mathcal{E}\{s \in S : f(p, s) \in p\}.$$

It is easy to see that $S_1(p)$ is a closed subsemigroup of the semigroup S.

Definition 4. We shall say that a point $p \in R$ satisfies condition O_i $(i = 1, 2, 3)$ if there exists a bicompact set $K \subseteq S$ such that condition K'_i $(i = 1, 2, 3)$ is fulfilled:

K'_1. $T_p = f(p, K)$.

K'_2. $T_p = f(p, K)$, and for any point $q \in T_p$ there exists an element $s \in S$ for which $p \in f(q, s)$.

K'_3. $T_p = f(q, K)$ for any point $q \in T_p$.

Condition O_2 (for $S = D^+$) was introduced by M. I. Minkevich (10). If a point p satisfies condition O_2, then the funnel T_p is called **closed** (10). Each of the conditions O_i $(i = 1, 2, 3)$ in the case of an ordinary dynamical system $(S = D)$ means periodicity of the motion issuing from the point p.

It is easy to see that O_3 implies O_2, and O_2 implies O_1. Examples show that O_2 is stronger than O_1. Let us show that O_3 is stronger than O_2.

Example 1. As the space R, take the set of points of three-dimensional space XYZ lying on the following curves: a) on the circles $x^2 + y^2 = 1,\ z = 0$ and $x^2 + y^2 = 4,\ z = 0$; b) on the spiral $x = (2 - e^{\varphi}) \sin \varphi,\ y = (2 - e^{\varphi}) \cos \varphi,\ z = 0$ $(-\infty < \varphi \leq 0,\ \varphi$ is the polar angle); c) on the semicircles lying in the half-plane $x = 0,\ z \geq 0$ and subtending the segments $[1;\ 2 - e^{-2\pi n}]$ $(n = 1, 2, ...)$ of the Y-axis as diameters; d) on the circle lying in the plane $x = 0$ and subtending the segment $[1, 2]$ of the Y-axis as diameter.

Prescribe along the curves lying in the plane $z = 0$ uniform motion in the positive direction with unit velocity, and along the curves lying in the plane $x = 0$ uniform motion with the same velocity, starting from the point $p = (0; 1; 0)$ in the direction of increasing z. In this way we define the p.n.o. (R, D^+, f). The point p satisfies condition O_2, but does not satisfy condition O_3.

The constructed example shows that Theorem 1 (§ 2) of the work of M. I. Minkevich (10) is false.

In the same work it is asserted (Theorem 8, § 2) that in a closed funnel (when $S = D^+$) the sections repeat periodically, starting from some time $T \geq 0$. The following example shows that the indicated theorem is also false.

Example 2. Let the circle L_1 have length π, and let the circle L_2 have length 1, lie in the same plane, and touch the circle L_1 from within at the point p. Prescribe on these circles uniform motion in one and the same direction with unit velocity. In the resulting system the point p satisfies condition O_3, but the sections do not repeat, owing to the incommensurability of the numbers 1 and π.

However, the following theorem holds.

Theorem 5. Let $S = D^+$, $p \in R$. The sections of the funnel T_p repeat periodically, starting from some $T \geq 0$, if at least one of the following conditions is satisfied:

1) the point p satisfies condition O_3, and the semigroup $S_1(p)$ is multigenic [(11)];

2) the point p satisfies condition O_1 and $S_1(p)$ contains a nonempty interval.

4°. Consider the following property of a point p, analogous to the property of Poisson stability:

P. For any neighborhood $U(p)$ and any point $q \in T_p$, there exists an element $s \in S$ such that $f(q, s) \cap U(p) \neq \Lambda$.

Theorem 6. In a transitive set all points satisfy condition P.

For nontransitive minimal bicompact sets the assertion of Theorem 6, generally speaking, does not hold.

Definition 5. We shall say that a semi-invariant set Σ is **regionally transitive** [(5)], if for any two open in Σ sets U and V there exists an element $s \in S$ such that $U \cap f(V, s) \neq \Lambda$.

A transitive set is regionally transitive. If the funnel T_p of every point $p \in \Sigma$ contains an interior (relative to Σ) point and the set Σ is regionally transitive, then Σ is transitive.

There exist regionally transitive bicompact minimal semi-invariant sets that are not transitive.

Definition 6. A point $p \in R$ will be called **orbitally stable** if for any neighborhood $\alpha \in U$ there exists a neighborhood $\beta \in U$ such that from $q \in \beta(p)$ it follows that $T_q \subseteq \alpha(T_p)$.

Theorem 7. If all points of the set T_p are points of unrest and at the same time points of orbital stability, and the point p satisfies condition P, then $\overline{T}_p$ is a transitive set.

Theorem 8. In order that a semi-invariant bicompact set Σ be transitive, it is necessary and sufficient that it be regionally transitive and that all points $p \in \Sigma$ be orbitally stable.

There exist bicompact semi-invariant nontransitive sets all of whose points are orbitally stable.

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Note: Figure translations are in progress. See original paper for figures.