Mathematics

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Abstract

Full Text

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ON A DIOPHANTINE EQUATION OF THE THIRD DEGREE

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The present article is a continuation of the work ([1]). Here we give a formula for finding all primitive solutions, i.e., solutions in relatively prime numbers, of a Diophantine equation of the third degree, and also an asymptotic formula for the number of primitive integral points lying on a certain two-dimensional surface of the third order in four-dimensional space.

Consider the Diophantine equation

$$\begin{vmatrix} x & y & z \\ -7z & x+7z & y \\ -7y & 7y-7z & x+7z \end{vmatrix} = t^3 \tag{1}$$

and let us find all its primitive solutions, i.e., solutions for which $\gcd(x, y, z, t) = 1$. Define the following four forms in three variables m, n, and l:

$$\begin{aligned} X_1(m,n,l) &= 98l^3 + m^3 - 42mn^2 + 98ml^2 - 14m^2n + 28m^2l - 28n^3 \\ &\quad - 49n^2l + 49nl^2, \\ X_2(m,n,l) &= -49l^3 - 35ml^2 - 7m^2l + 6m^2n + 14mn^2 + 14mnl \\ &\quad + 7n^3 + 28n^2l, \\ X_3(m,n,l) &= 7n^3 - 7nl^2 + 7n^2l + 9mn^2 - 3m^2l - 7ml^2 + 3m^2n, \\ X_4(m,n,l) &= 49l^3 + m^3 - 7mn^2 + 49ml^2 + 14m^2l + 21mnl - 7n^3 + 49nl^2. \end{aligned}$$

For integral m, n, l, put

$$D = \gcd(X_1(m,n,l),\ X_2(m,n,l),\ X_3(m,n,l)).$$

Theorem 1. *All primitive solutions of the Diophantine equation* (1) *are obtained exactly once from the formulas*

$$x = \frac{X_1(m,n,l)}{D}, \qquad y = \frac{X_2(m,n,l)}{D}, \qquad z = \frac{X_3(m,n,l)}{D}, \qquad t = \frac{X_4(m,n,l)}{D},$$

where m, n, l run through all triples of integers subject to the condition $\gcd(m, n, l) = 1$.

Proof. We shall only outline the proof.

Lemma 1. *All solutions in rational numbers of the equation*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ -7x_3 & x_1 + 7x_3 & x_2 \\ -7x_2 & 7x_2 - 7x_3 & x_1 + 7x_3 \end{vmatrix} = 1 \tag{2}$$

are obtained exactly once when, in the formulas

$$x_1 = \frac{X_1(m,n,l)}{X_4(m,n,l)}, \qquad x_2 = \frac{X_2(m,n,l)}{X_4(m,n,l)}, \qquad x_3 = \frac{X_3(m,n,l)}{X_4(m,n,l)},$$

the parameters m, n, l run through all integers satisfying the conditions:

1) *either* $m > 0,\ \gcd(m, n, l) = 1$*;*

2) *or* $m = 0,\ n > 0,\ \gcd(m, n, l) = 1$*;*

3) $m = 0,\ n = 0,\ l = 1$.

Lemma 2.

$$\gcd(X_1(m,n,l),\ X_2(m,n,l),\ X_3(m,n,l))$$

$$= \gcd(X_1(m,n,l),\ X_2(m,n,l),\ X_3(m,n,l),\ X_4(m,n,l)).$$

Without proving these lemmas, let us only note that in proving them we used the close connection of equation (2) with the cyclic cubic extension $R(\alpha)$ of the field of rational numbers R (α is a root of the equation irreducible over R, $x^3 - 7x + 7 = 0$).

It is clear that our theorem follows from the indicated lemmas.

Remark. The quantity D occurring in the formulation of the theorem can be given another interpretation. Denote by Ω the ring of integers of the field $R(\alpha)$. Let $B = m + n\alpha + l\alpha^2$ be an integer of the field $R(\alpha)$, with $\gcd(m, n, l) = 1$. The decomposition of the ideal (B) (if B is not a unit of the ring Ω) into prime ideals can be written in the form

$$(B) = (\pi_7)^{\nu} \prod_s (\pi_s)^{k_s} (\sigma\pi_s)^{k'_s},$$

where $\nu = 0, 1$, or 2, $k_s > 0$, $k'_s \geq 0$, and σ is a generating automorphism of the Galois group of the field $R(\alpha)$. Then

$$D = \begin{cases} 1, & \text{if } B \text{ is a unit of the ring } \Omega, \\ 7^{\nu} \prod\limits_{\substack{s \\ k'_s > 0}} (\operatorname{Norm} \pi_s)^{\min(k_s, 2k'_s)}, & \text{if } B \text{ is not a unit of the ring } \Omega. \end{cases}$$

From this remark it is seen that D may grow without bound as m, n, l vary. This circumstance makes the study of the distribution of integral points on the surface (1) difficult. However, for one case we can solve the corresponding distribution problem.

Denote by $F(h)$ the number of primitive integral points of the form (l_1, l_2, l_3, l_4) with $l_4 > 0$, lying on the surface given parametrically by

$$x_1 = 14y_1^3 + 7y_1^2 y_2 - 7y_1 y_2^2 - 4y_2^3,$$

$$x_2 = -7y_1^3 + 4y_1^2 y_2 + y_2^3,$$

$$x_3 = -y_1^2 y_2 + y_1 y_2^2 + y_2^3,$$

$$x_4 = 7y_1^3 + 7y_1^2 y_2 - y_2^3,$$

where y_1 and y_2 run through real numbers, $y_2 > 0$, and $y_1^2 + y_2^2 \leq h$.

Theorem 2. As $h \to \infty$,

$$F(h) = \frac{\theta}{16\pi^2} h + O(\sqrt{h} \ln h),$$

where

$$\theta = \frac{48\sqrt[3]{49} - 1}{7} \operatorname{arctg} \frac{7\,[k_3(k_2 - k_1) + 49 + k_1 k_2]}{(49 + k_1 k_2)k_3 - 49(k_2 - k_1)} + 49 \operatorname{arctg} \frac{k_3(k_2 - k_1) + k_1 k_2 + 1}{k_3(1 + k_1 k_2) + k_1 - k_2}.$$

Here k_1, k_2, k_3 are the roots of the equation

$$7z^3 + 7z^2 - 1 = 0, \qquad k_1 < 0, \quad k_2 < 0, \quad k_3 > 0, \quad |k_1| > |k_2|.$$

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REFERENCES

1. V. P. Myakishev, DAN, **143**, No. 4, 785 (1962).

Note: Figure translations are in progress. See original paper for figures.