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Abstract

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THEORY OF ELASTICITY

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ON THE FUNDAMENTAL EQUATIONS OF THE THEORY OF ELASTIC SHELLS

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In dynamic problems of the theory of elastic shells, the self-adjointness of the initial differential equations of equilibrium plays an essential role. Refining the equations of Love–Timoshenko, V. Z. Vlasov (^{1}) obtained self-adjoint equilibrium equations for a circular cylindrical shell and for a spherical shell. For shells of arbitrary configuration, self-adjoint equilibrium equations can be obtained by generalizing the method of initial functions developed by V. Z. Vlasov (^{2}) as applied to the problem of equilibrium of a thick plate; however, in the general case this method leads to extremely cumbersome calculations, especially in the presence of body forces, which are unavoidable in dynamic problems.

In the present article we indicate a method for solving the three-dimensional problem of the theory of elasticity for a body having the form of a thin shell, by means of which one can relatively simply construct a self-adjoint system of differential equilibrium equations for a shell of arbitrary configuration and variable thickness, without invoking any simplifying hypotheses. This method is analogous to the method presented by us in the article (^{3}).

Using the system of orthogonal curvilinear coordinates α, β, γ customary in shell theory, we introduce into consideration the vectors

$$\begin{aligned}\mathbf{F}_j(\alpha,\beta) = \int_{-h/2}^{h/2} \mathbf{P}\gamma^j(1+k_1\gamma)(1+k_2\gamma)\,d\gamma + \mathbf{p}_e\left[\gamma^j(1+k_1\gamma)(1+k_2\gamma)\right]_{\gamma=h/2} \\ + \mathbf{p}_i\left[\gamma^j(1+k_1\gamma)(1+k_2\gamma)\right]_{\gamma=-h/2}, \qquad j=0,1,\dots,\end{aligned} \tag{1}$$

where $\mathbf{P}$, $\mathbf{p}_e$, and $\mathbf{p}_i$ are vectors determining the body and surface forces acting on the shell; $h(\alpha,\beta)$ is the shell thickness; $k_1(\alpha,\beta)$ and $k_2(\alpha,\beta)$ are the principal curvatures of the middle surface. The vector $\mathbf{F}_0$ determines the so-called reduced forces; the projection of the vector $\mathbf{F}_1$ onto the plane tangent to the middle surface determines the reduced moments.

We shall seek the projections of the elastic displacement $\mathbf{u}(\alpha,\beta,\gamma)$ in the form of power series

$$u_\alpha = \sum_{i=0}^{\infty} u_i(\alpha, \beta)\gamma^i, \qquad u_\beta = \sum_{i=0}^{\infty} v_i(\alpha, \beta)\gamma^i, \qquad u_\gamma = \sum_{i=0}^{\infty} w_i(\alpha, \beta)\gamma^i. \tag{2}$$

Using the fundamental equations of the theory of elasticity, transformed to curvilinear coordinates α, β, γ, one can express the projections of the vectors $\mathbf{P}$, $\mathbf{p}_e$, and $\mathbf{p}_i$ entering formulas (1) through the projections of the vector $\mathbf{u}$, and, in accordance with the expansions (2), construct for the projections $F_{j\alpha}, F_{j\beta}, F_{j\gamma}$ of the vectors $\mathbf{F}_j$, $j = 0, 1, \dots$, expansions in powers of the parameter h. In these expansions the coefficients of the powers of h will be differential expressions containing the functions u_i, v_i, w_i, $i = 0, 1, \dots$. Thus, the totality of the expansions obtained will constitute an infinite system of linear partial differential equations for the functions u_i, v_i, w_i, $i = 0, 1, \dots$, determining, in accordance with formulas (2), the desired elastic displacements $u_\alpha, u_\beta, u_\gamma$. Using the asymptotic method widely applied in shell theory

integrating linear differential equations containing a small parameter, one can construct asymptotic expansions for the functions u_i, v_i, w_i and thereby obtain, in accordance with formulas (2), asymptotic representations for the elastic displacements $u_\alpha, u_\beta, u_\gamma$.

Omitting in the infinite system of differential equations for the functions u_i, v_i, w_i all expressions having order of smallness higher than h^3, we obtain, for the functions u_i, v_i, w_i, $0 \le i \le 3$, a system of 12 equations determining the elastic displacements $u_\alpha, u_\beta, u_\gamma$ with errors of order h^4. Eliminating from these equations the functions u_i, v_i, w_i, $i = 1, 2, 3$, one can construct for the functions u_0, v_0, w_0 a system of 3 equations determining these functions with errors of order h^4. Passing from the notation u_0, v_0, w_0 to the generally accepted notation u, v, w and, for simplicity, restricting ourselves to the case of a homogeneous shell of constant thickness, these 3 equations can be represented in the form

$$
\begin{aligned}
&\frac{1}{A}\left\{1-\frac{h^2}{8}\left[2H^2-(1+\nu)K\right]\right\}\frac{\partial\varphi_1}{\partial\alpha}+(1-\nu)\left[1+\frac{h^2}{24}k_1(2k_1+k_2)\right]\psi_2\\
&\quad+\frac{h^2}{24A}\frac{\partial}{\partial\alpha}\left[K_1\varphi_1+3H\varphi_2-\frac{2(2+\nu)}{1-\nu}H\varphi_3+3(1+\nu)\varphi_4+\frac{1+2\nu}{1-\nu}\varphi_5\right]\\
&\quad-\frac{h^2(1-\nu)}{12B}\frac{\partial}{\partial\beta}\left[(4H^2-K)\psi_1-4H\psi_4+\psi_5\right]\\
&\quad+\frac{h^2(2k_1+3k_2)}{24A}\frac{\partial}{\partial\alpha}\left(\frac{1-4\nu}{1-\nu}H\varphi_1-\frac{1-\nu}{2}\varphi_2+\varphi_3\right)\\
&\quad-\frac{h^2(1-\nu)k_2}{24A}\frac{\partial\varphi_3}{\partial\alpha}+\frac{h^2(1-\nu)k_1}{3B}\frac{\partial(H\psi_1-\psi_4)}{\partial\beta}+\frac{1-\nu^2}{Eh}X=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{B}\left\{1-\frac{h^2}{8}\left[2H^2-(1+\nu)K\right]\right\}\frac{\partial\varphi_1}{\partial\beta}+(1-\nu)\left[1+\frac{h^2}{24}k_2(k_1+2k_2)\right]\psi_3\\
&\quad+\frac{h^2}{24B}\frac{\partial}{\partial\beta}\left[K_1\varphi_1+3H\varphi_2-\frac{2(2+\nu)}{1-\nu}H\varphi_3+3(1+\nu)\varphi_4+\frac{1+2\nu}{1-\nu}\varphi_5\right]\\
&\quad+\frac{h^2(1-\nu)}{12A}\frac{\partial}{\partial\alpha}\left[(4H^2-K)\psi_1-4H\psi_4+\psi_5\right]\\
&\quad+\frac{h^2(3k_1+2k_2)}{24B}\frac{\partial}{\partial\beta}\left(\frac{1-4\nu}{1-\nu}H\varphi_1-\frac{1-\nu}{2}\varphi_2+\varphi_3\right)\\
&\quad-\frac{h^2(1-\nu)k_1}{24B}\frac{\partial\varphi_3}{\partial\beta}-\frac{h^2(1-\nu)k_2}{3A}\frac{\partial(H\psi_1-\psi_4)}{\partial\alpha}+\frac{1-\nu^2}{Eh}Y=0,
\end{aligned}
$$

$$
\begin{aligned}
&-2H\varphi_1+(1-\nu)\varphi_2+\frac{h^2}{24}\Bigg\{HK_2\varphi_1-\left[2H^2-(1-\nu)K\right]\left[3\varphi_2-\frac{2(2+\nu)}{1-\nu}\varphi_3\right]\\
&\qquad-6(1+\nu)H\varphi_4-\frac{2(1+2\nu)}{1-\nu}H\varphi_5\Bigg\}\\
&\quad+\frac{h^2}{24AB}\frac{\partial}{\partial\alpha}\left\{(1-\nu)(2k_1+k_2)B\psi_2-\frac{B}{A}\left[\frac{2(2+\nu)}{1-\nu}\frac{\partial(H\varphi_1)}{\partial\alpha}-3(1+\nu)k_2\frac{\partial\varphi_1}{\partial\alpha}+(1-\nu)\frac{\partial\varphi_2}{\partial\alpha}-2\frac{\partial\varphi_3}{\partial\alpha}\right]\right\}\\
&\quad+\frac{h^2}{24AB}\frac{\partial}{\partial\beta}\left\{(1-\nu)(k_1+2k_2)A\psi_3-\frac{A}{B}\left[\frac{2(2+\nu)}{1-\nu}\frac{\partial(H\varphi_1)}{\partial\beta}-3(1+\nu)k_1\frac{\partial\varphi_1}{\partial\beta}+(1-\nu)\frac{\partial\varphi_2}{\partial\beta}-2\frac{\partial\varphi_3}{\partial\beta}\right]\right\}\\
&\quad+\frac{1-\nu^2}{Eh}Z=0,
\end{aligned}
\tag{3}
$$

where $A(\alpha,\beta)$ and $B(\alpha,\beta)$ are the Lamé coefficients for the middle surface of the shell, E is the modulus of elasticity, ν is Poisson' s ratio,

$$
H=\frac{k_1+k_2}{2},\qquad K=k_1k_2,\qquad K_1=\frac{2}{1-\nu}\left[\frac{3\nu(4-\nu)}{1-\nu}H^2-(1+\nu)(1+2\nu)K\right],
$$

$$K_2 = \frac{6(1+2\nu)}{1-\nu} K - 12H^2 - 2K_1, \tag{4}$$

and φ_j and ψ_j, $j = 1, 2, \dots, 5$, are differential expressions determined by the formulas

$$\varphi_1 = \frac{1}{AB}\left[\frac{\partial(Bu)}{\partial\alpha} + \frac{\partial(Av)}{\partial\beta}\right] + 2Hw,$$

$$\varphi_2 = \frac{1}{AB}\left[\frac{\partial(k_2 Bu)}{\partial\alpha} + \frac{\partial(k_1 Av)}{\partial\beta}\right] + 2Kw,$$

$$\varphi_3 = \frac{1}{2AB}\left\{\frac{\partial}{\partial\alpha}[(2k_1+3k_2)Bu] + \frac{\partial}{\partial\beta}[(3k_1+2k_2)Av]\right\} -$$

$$-\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial w}{\partial\beta}\right)\right] + 3Kw,$$

$$\varphi_4 = \frac{1}{AB}\left[\frac{\partial(KBu)}{\partial\alpha} + \frac{\partial(KAv)}{\partial\beta}\right] - \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{k_2 B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{k_1 A}{B}\frac{\partial w}{\partial\beta}\right)\right],$$

$$\varphi_5 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial\varphi_1}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial\varphi_1}{\partial\beta}\right)\right],$$

$$\psi_1 = \frac{1}{2AB}\left[\frac{\partial(Bv)}{\partial\alpha} - \frac{\partial(Au)}{\partial\beta}\right], \tag{5}$$

$$\psi_2 = Ku - \frac{1}{B}\frac{\partial\psi_1}{\partial\beta} - \frac{k_2}{A}\frac{\partial w}{\partial\alpha},$$

$$\psi_3 = Kv + \frac{1}{A}\frac{\partial\psi_1}{\partial\alpha} - \frac{k_1}{B}\frac{\partial w}{\partial\beta},$$

$$\psi_4 = \frac{1}{2AB}\left[\frac{\partial(k_2 Bv)}{\partial\alpha} - \frac{\partial(k_1 Au)}{\partial\beta}\right],$$

$$\psi_5 = \frac{1}{4AB}\left\{\frac{\partial}{\partial\alpha}[k_2(k_1+2k_2)Bv] - \frac{\partial}{\partial\beta}[k_1(2k_1+k_2)Au]\right\} -$$

$$-\frac{1}{4AB}\left\{\frac{\partial}{\partial\alpha}\left[(k_1+2k_2)\frac{\partial w}{\partial\beta}\right] - \frac{\partial}{\partial\beta}\left[(2k_1+k_2)\frac{\partial w}{\partial\alpha}\right]\right\} +$$

$$+\frac{1}{4AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial\psi_1}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial\psi_1}{\partial\beta}\right)\right].$$

Equations (3) differ from the known differential equations of shell theory in the structure of the differential expressions containing the factor h^2. The functions $X(\alpha,\beta)$, $Y(\alpha,\beta)$, $Z(\alpha,\beta)$ occurring in equations (3), with errors of order h^2, are determined by the formulas

$$X=F_{0\alpha}+k_1F_{1\alpha}+\frac{\nu}{(1-\nu)A}\frac{\partial F_{1\gamma}}{\partial\alpha},$$

$$Y=F_{0\beta}+k_2F_{1\beta}+\frac{\nu}{(1-\nu)B}\frac{\partial F_{1\gamma}}{\partial\beta}, \tag{6}$$

$$Z=F_{0\gamma}+\frac{1}{AB}\left[\frac{\partial(BF_{1\alpha})}{\partial\alpha}+\frac{\partial(AF_{1\beta})}{\partial\beta}\right]-\frac{2\nu}{1-\nu}HF_{1\gamma}.$$

The system of differential equations (3) is self-adjoint and can serve as the basis for calculations in the investigation of vibrations of a thin-walled homogeneous elastic shell of constant thickness. For an open shell, the boundary conditions determined by the forces acting on the shell edges, or by the constraints imposed on the elastic displacements by one or another fixing of these edges, should be appended to the differential equations (3).

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CITED LITERATURE

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[2] V. Z. Vlasov, *Izv. AN SSSR, OTN*, No. 7 (1955).
[3] I. M. Rapoport, *DAN*, **153**, No. 4 (1963).

Note: Figure translations are in progress. See original paper for figures.

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