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Abstract

Full Text

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On a Universal Bicompactum of Weight $\aleph$

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1. According to a well-known theorem of P. Aleksandrov, the Cantor discontinuum $\Delta_{\aleph_0}$ has the universality property in the class of bicompacta of weight $\leqslant \aleph_0$, consisting in the fact that every bicompactum of weight $\leqslant \aleph_0$ is its continuous image. P. Aleksandrov also gave a simple topological definition of $\Delta_{\aleph_0}$ as a zero-dimensional perfect bicompactum of weight $\aleph_0$. In (^{1}) A. Esenin-Volpin, using the generalized continuum hypothesis, proved the existence, in the same sense, of a universal bicompactum for any weight $\mathfrak{m}$.

In the present paper it will be proved (using the continuum hypothesis, which below is always assumed unless the contrary is stipulated) that the Čech remainder on the natural number series

$$\Delta_{\aleph} = \beta N \setminus N$$

is a universal bicompactum for the class of bicompacta of weight $\leqslant \aleph$, and a topological definition of $\Delta_{\aleph}$ will also be given and a theorem for $\Delta_{\aleph}$ obtained that is completely analogous to the theorem for $\Delta_{\aleph_0}$.

2. Below we use the following notation in the Boolean algebra L: $a+b$, ab, b', $ab' = a \setminus b = a - b$ (the latter when $b \leqslant a$); the least element is denoted by 0; the principal ideal $\{x \mid x \leqslant a\}$ is denoted by L_a, the dual principal ideal $\{x \mid x \geqslant a\}$ by L^a. As Nowak showed (^{2}), the Boolean algebra of all open-closed sets of $\Delta_{\aleph}$ is isomorphic to the Boolean algebra $L_{\aleph}$, defined as follows.

As elements of $L_{\aleph}$ one takes equivalence classes in the system of all subsets of the natural number series, where $E \sim G$ if $(E \setminus G) \cup (G \setminus E)$ is finite, and for $e, g \in L_{\aleph}$ one has $e < g$ if $G \setminus E$ is infinite and $E \setminus G$ is finite, $E \in e$, $G \in g$.

We shall say that in a partially ordered set T the following hold: the **simplest separability**, if for any $e < h$ from T there exists g such that $e < g < h$; **Cantor separability**, if for any set

$$e_1 < \cdots < e_n < \cdots < h$$

of type $\omega + 1$ there exists g such that $e_n < g < h$; and **du Bois-Reymond separability**, if for any set

$$e_1 < \cdots < e_n < \cdots < h_n < \cdots < h_1$$

of type $\omega + \omega^*$ there exists g such that $e_n < g < h_n$. We shall say that a zero-dimensional bicompactum has the corresponding separability if the Boolean algebra of its open-closed sets has it. In particular, the simplest separability is equivalent to the requirement that the bicompactum be perfect. The remainder $\Delta_\aleph$, besides the simplest separability, also has the Cantor and du Bois-Reymond separabilities, as is testified (in view of Nowak's isomorphism) by the theorems of the same name on subsets of the natural numbers (^{3}, p. 715). However, for zero-dimensional bicompacta of weight $\aleph$, in general, none of the three separabilities implies another, and no two imply the third. Indeed, from Cantor separability it follows that the intersection of a strictly decreasing sequence of open-closed sets contains an interior point, but the bicompactum

$$\left[\bigcup_n e_n\right](\Delta_\aleph) \quad (e_1 \subset e_2 \subset \cdots \text{ and open-closed in } \Delta_\aleph)$$

does not have this property, although the other two separabilities hold for it.

From du Bois-Reymond separability it follows that in the bicompactum there are no cappa-

points ((4), p. 912), while the ordered bicompactum of type $2^{\omega_1^*}$ (notations from (5)) contains them, although it satisfies the other two separation axioms.

Theorem 1. *Every perfect zero-dimensional bicompactum of weight $\aleph$ with the separation axioms of Cantor and of Du Bois-Reymond is homeomorphic to $\Delta_\aleph$, and, independently of the continuum hypothesis, $\Delta_\aleph$ is continuously mapped onto any bicompactum of weight $\leq \aleph_1$.*

Lemma 1. *In a Boolean algebra L with Cantor' s separation axiom, every ideal cofinal with no more than a countable set is closed (see (6), p. 95).*

Let A be an ideal in L, cofinal with the sequence

$$a_1 < \cdots < a_n < \cdots.$$

In what follows we shall regard L as the family of all open-closed sets of some zero-dimensional bicompactum Δ, while preserving, however, the structural notation. Let $a \leq b$ for every $b \supset \bigcup_n a_n$ and a fixed $a > 0$, and moreover for all a_n let $a \nleq a_n$. The intersection of all the indicated b' s is

$$\left[\bigcup_n a_n\right](\Delta).$$

Consequently,

$$a \subseteq \left[\bigcup_n a_n\right],$$

and since a is open, we have

$$a \cap \left(\bigcup_n a_n\right) = \Lambda$$

and $aa_n > 0$ for $n \geq n_0$. But, by assumption, $a \setminus a_i > 0$ and $a \setminus a_n$ decrease. If $\{a \setminus a_n\}$ does not stabilize, then, by Cantor’ s separation axiom, there exists a c such that

$$0 < c < a \setminus a_n;$$

hence

$$0 < c \leq a_n \subseteq \left[\bigcup_n a_n\right]$$

and

$$c \cap \left(\bigcup_n a_n\right) = \Lambda,$$

which is impossible. The stabilization case does not require use of the separation axiom.

Lemma 2. *If in a Boolean algebra L with the separation axioms of Cantor and of Du Bois-Reymond an ideal A is cofinal with no more than a countable set and $C = \{c_i\}$ is a set of no more than countably many elements in $L \setminus A$, then there exists a principal ideal L_g such that $A \subseteq L_g$ and $C \subseteq L \setminus L_g$.*

Let A be cofinal with

$$a_1 < \cdots < a_n < \cdots .$$

By Lemma 1, for each c_i choose e_i such that $A \leq e_i$ and $c_i \nleq e_i$. Put

$$g_j = \bigwedge_{i=1}^{j} e_i;$$

then

$$g_1 \geq \cdots \geq g_j \geq \cdots ,$$

with

$$g_j \leq e_i \quad (i = 1, \dots , j),$$

so that

$$c_i \nleq g_j \quad (i = 1, \dots , j).$$

If $\{g_j\}$ stabilizes at g_{j_0}, then $L_{g_{j_0}}$ is the required ideal. If, however, $\{g_j\}$ does not stabilize, then, by the Du Bois-Reymond separation axiom, there exists a g such that

$$a_n < g < g_j$$

and

$$c_i \nleq g;$$

then L_g is the required ideal.

Lemma 3. *Let in a Boolean algebra L with the separation axioms of the simplest kind, of Cantor, and of Du Bois-Reymond there be given three no more than countable sets*

$$A = \{a_l\}, \quad B = \{b_m\}, \quad C = \{c_n\},$$

with

$$a_1 < \dots < a_l < \dots < b_m < \dots < b_1,$$

and suppose that for any l, m, n *one has* $c_n \nleq a_l$ *and* $b_m \nleq c_n$*. Then there exists a* d *such that*

$$a_l < d < b_m$$

and d *is not comparable with any* c_n*.*

Let

$$\widetilde{A} = \bigcup_m \{x \mid x \leq a_l\}, \qquad \widetilde{B} = \bigcup_l \{x \mid x \geq b_m\};$$

then $\widetilde{A}$ is an ideal ($\widetilde{B}$ is a dual ideal), cofinal (coinitial) with no more than a countable set, and

$$\widetilde{A} < \widetilde{B}.$$

Using the separation axioms of Cantor and Du Bois-Reymond, we find such g_0 and g_1 that

$$\widetilde{A} \leq g_0 < g_1 \leq \widetilde{B},$$

and, by Lemma 2 and the dual proposition to it, choose such h_0 and h_1 that

$$\widetilde{A} \subseteq L_{h_1}, \qquad \widetilde{B} \subseteq L^{h_1}, \qquad C \subseteq L \setminus (L_{h_0} \cup L^{h_1}).$$

Let

$$t_0 = g_0 h_0, \qquad t_1 = g_1 + h_1;$$

then

$$\widetilde{A} \leq t_0 < t_1 \leq \widetilde{B}, \qquad C \subseteq L \setminus (L_{t_0} \cup L^{t_1}).$$

Put

$$c_n^0 = c_{nt}1;$$

since

$$c_n \in L^{t_1},$$

we have

$$c_n^0 < t_1.$$

Let

$$e_p = \bigvee_{n=1}^{p} c_n^0,$$

and put

$$q_1 = e_1,$$

$$q_2 = q_1 + (e_2 - q_1), \dots, \qquad q_p = q_{p-1} + (e_p - q_{p-1}), \dots, \qquad r_0 = t_0, \dots,$$

$$\dots, \ r_p = t_0 + q_p, \dots, \qquad t_0 = r_0 \leq r_1 \leq r_2 \leq \dots < t_1.$$

Using the separation axiom of the simplest kind or Cantor' s axiom, we find an r such that

$$r_p < r < t_1 \quad (p = 0, 1, \dots),$$

and then

$$d = t_0 + (t_1 - r)$$

is the required element.

Now Theorem 1 is proved analogously to Rudin' s theorem in (7), 4.7, on homeomorphisms of $\Delta_\aleph$ onto itself, but as applied to two Boolean algebras and with replacement of Lemma 4.8 from (7) by our Lemma 3 (cf. also (1)).

Corollary. The class of bicompacta of weight $\leq \aleph$ coincides with the class of all bicompact extensions of the natural row.

2. Let $C_\aleph = \{y\}$ be the lexicographically ordered set of all sequences of type ω_1 of real numbers y_ξ, $0 \leq y_\xi \leq 1$, and let $I_\aleph$ be the ordered set obtained after removing from $C_\aleph$ all kappa-points.

Theorem 2 is analogous to the theorem on the universality of the Baire 0-space.

Theorem 2. The ordered space $I_\aleph$ is condensed onto $\Delta_\aleph$ and therefore is mapped continuously onto any bicompactum of weight $\leq \aleph$.

Since $\Delta_\aleph$ is a condensation of $T^1\Delta_\aleph$ (^{8}), it is enough for us to prove that $I_\aleph$ and $T^1\Delta_\aleph$ are homeomorphic. Let $\mathfrak{G} = \{\Gamma_\nu\}_{\nu<\omega_1}$ be an enumerated collection of all nonempty open-closed sets of $\Delta_\aleph$; here we assume that, if ζ is zero or limit, then $\Gamma_{\zeta+2k}$ and $\Gamma_{\zeta+2k+1}$ are complements in $\Delta_\aleph$ $(0 \leq k < \omega)$. Let $x = \{i_\xi\}_{\xi<\omega_1}$ be a sequence of 0' s and 1' s.

Define $\mathfrak{D}(x) = \{D_\xi(x)\}$ by induction: $D_0(x) = \Gamma_0$ for $i_0 = 0$, and $D_0(x) = \Gamma_1$ for $i_0 = 1$; if $D_\xi(x)$ are defined for $\xi < \eta$, then for limit η

$$D_\eta(x) = \bigcap_{\xi<\eta} D_\xi(x),$$

and for $\eta = \eta_0 + 1$,

$$D_\eta(x) = D_{\eta_0}(x) \cap \Gamma_{\nu_0},$$

if $i_\eta = 0$, and

$$D_\eta(x) = D_{\eta_0}(x) \cap \Gamma_{\nu_0+1},$$

if $i_\eta = 1$, where $\nu_0 = \nu_0(\eta)$ is the least of those ν for which simultaneously

$$D_{\eta_0} \cap \Gamma_\nu \supset \Lambda, \qquad D_{\eta_0} \cap \Gamma_{\nu+1} \supset \Lambda.$$

It is clear that $\nu_0(\eta)$ strictly increases. The sequence $\mathfrak{D}(x)$ decreases and has a nonempty intersection by virtue of the bicompactness of $\Delta_\aleph$, and from the construction it follows easily that this intersection contains a single point; and hence we shall write $\Delta_\aleph = \{x\}$. Let us prove that $\mathfrak{D} = \bigcup_x \mathfrak{D}(x)$ forms an open base in $T^1\Delta_\aleph$.

Indeed, let

$$Ox = \bigcap_{n<\omega} G_n(x),$$

where $G_n(x)$ are neighborhoods of x in $\Delta_\aleph$. Since

$$\bigcap_n D_{\xi_n}(x) = x \in G_n(x),$$

then, by the bicompactness of $\Delta_\aleph$, there exists ξ_n such that

$$D_{\xi_n}(x) \subseteq \bigcap G_n(x);$$

taking $\xi' > \xi_n$, we have

$$D_{\xi'}(x) \subseteq \bigcap_n D_{\xi_n}(x) \subseteq \bigcap_n G_n(x) = Ox.$$

Now enumerate all even limit numbers: $\tau_0 = \omega,\ \tau_1 = \omega 2, \dots, \tau_\pi, \dots$, and let

$$\mathfrak{D}_\pi = \{D_{\tau_\pi}(x) \mid x \in \Delta_\aleph\};$$

then

$$\mathfrak{D} = \bigcup_{\pi<\omega_1} \mathfrak{D}_\pi,$$

where $\{\mathfrak{D}_\pi\}$ is a sequence of disjoint open-closed covers of $T^1\Delta_\aleph$, and each element of $\mathfrak{D}_\pi$ contains $\aleph$ elements of $\mathfrak{D}_{\pi+1}$. Let $C(y_0, \dots, y_\xi, \dots)$ be the set of all points of $C_\aleph$ beginning with the complex

$$(y_0, \dots, y_\xi, \dots)_{\xi<\eta<\omega_1},$$

and let

$$\mathfrak{C}_\eta = \{C(y_0, \dots, y_\xi, \dots)\};$$

then $\{\mathfrak{C}_\eta\}_{\eta<\omega_1}$ is a sequence of covers of $C_\aleph$ by disjoint systems of segments, while the order type of $\mathfrak{C}_\eta$ is equal to 0^{η^*}, and therefore the kappa-points of $C_\aleph$, and only they, are the ends of segments from

$$\bigcup_{\eta<\omega_1} \mathfrak{C}_\eta.$$

Removing them, we obtain a sequence of interval covers of $I_\aleph$, and the joining of the latter will give a base of $I_\aleph$, isomorphic to $\mathfrak{D}$, whence the required homeomorphism follows.

We shall say that M is of the $\aleph$-category in S if M is (not) representable as the union of $\leq \aleph$ nowhere dense sets in S. It is easy to see that $\Delta_\aleph$ is of the $\aleph$-category, and if Π is the set of all P-points of $\Delta_\aleph$ (7), then $\Delta_\aleph \setminus \Pi$ is of the $\aleph$-category, since $\Delta_\aleph \setminus \Pi$ is the union of the boundaries of a system of $\aleph^{\aleph_0} = \aleph$ all countable intersections of open-closed sets of $\Delta_\aleph$. By modifying the proof of Theorem 2 one obtains

Theorem 3. The set of all P-points of $\Delta_\aleph$ is homeomorphic to $I_\aleph$, and therefore every ordered space of cardinality up to the set I is of the $\aleph$-category.

3. Theorem 4. All maximal ordered subsets of $L_\aleph$ are similar and have type $1 + \eta_1 + 1$, where η_1 is the normal Hausdorff type ((9), p. 181).

This theorem follows at once from the separability of $L_\aleph$ and Theorem II of (9), p. 181. Theorem 4 gives a concrete embodiment of several somewhat indefinite thoughts of Luzin on the "Pythagoras phenomenon" on the "transfinite line" (3), p. 721. In particular, since the Dedekind completion of a set of type $1+\eta_1+1$ is like $C_\aleph$ (this also follows easily from the cited theorem of Hausdorff), and $C_\aleph$ has cardinality $2^\aleph$, there turn out to be more of Luzin' s "transfinite irrationalities," as of the irrationalities of the ordinary line, than of "real" (rational) points: they correspond to Dedekind cuts in a set of type $1 + \eta_1 + 1$.

At the same time one also obtains an immediate solution of all Luzin' s problems from (3), p. 721, which, however, is not new (2).

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Note: Figure translations are in progress. See original paper for figures.

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