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Abstract

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ON THE DEFORMATION OF COMPLEX STRUCTURES OF ALGEBRAIC MANIFOLDS

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A complex-analytic family of structures is a triple $(\mathcal{V}; M, \pi)$, where π is a holomorphic mapping of the complex manifold $\mathcal{V}$ onto the complex manifold M, which is a smooth locally trivial fibration (1). This means, in particular, that the inverse image V_t under the mapping π of each point $t \in M$ is diffeomorphic to one and the same manifold X.

Consequently, the restriction $\pi^{-1}(U)$ of the fibration $\mathcal{V}$ over any contractible neighborhood $U \subset M$ is diffeomorphic to the direct product $X \times U$, and for any $t \in U$ the homomorphisms of cohomology groups

$$i_t^*: \quad H^i(\pi^{-1}(U), Z) \to H^i(V_t, Z),$$

induced by the embedding $i_t : V_t \to \pi^{-1}(U)$, are isomorphisms.

1. In the first part of the note we study the set $A \subset M$ of those points $t \in M$ for which V_t is an algebraic manifold. By Kodaira' s theorem (2), a compact complex manifold V is algebraic if and only if it admits a Hodge metric. Let $t_0 \in A$, and let $c_0 \in H^2(V_{t_0}, Z)$ be the cohomology class to which belongs the differential form ω_{t_0} of type $(1, 1)$, associated with the Hodge metric on the manifold V_{t_0}. We shall determine for which $t \in M$ the harmonic form ω_t on the manifold V_t, belonging to the class

$$c_t = i_t^* i_{t_0}^{*-1} c_0,$$

is associated with some Hodge metric. Obviously, for this it is necessary that the form ω_t be of type $(1, 1)$, i.e. $\Pi^{0,2}\omega_t = 0$. As usual, $\Pi^{p,q}\omega$ denotes the component of type (p, q) of the form ω. If t belongs to a sufficiently small neighborhood U of the point t_0, then the condition $\Pi^{0,2}\omega_t = 0$ is also sufficient by virtue of the continuity in t of the form ω_t.

Let $k_1(t), \dots, k_l(t)$ be a basis of the space of holomorphic two-forms on the manifold V_t. Then

$$\Pi^{0,2}\omega_t = \overline{\Pi^{2,0}\omega_t} = \sum F_i(t)\,\overline{k_i(t)},$$

where $F_i(t)$ are functions on the manifold M.

Proposition 1. The functions $F_i(t)$ are holomorphic in the variables t for a suitable choice of the family of bases $k_i(t)$.

Thus, the following assertion concerning the structure of the set A proves to be true.

Theorem 1. Let $(\mathcal{V}, M, \pi)$ be a family of complex structures, let $V_{t_0} = \pi^{-1}(t_0)$ be an algebraic manifold, and let $h^{2,0}$ be the dimension of the space of holomorphic two-forms on V_{t_0}. Then in some neighborhood $U \subset M$ of the point t_0 there exists an analytic submanifold $B \subset U$ of codimension not exceeding $h^{2,0}$, such that for all $t \in B$ the manifold V_t is algebraic (i.e. $B \subset A$). Moreover, the set $A \subset M$ of all points $t \in M$ such that

such that V_t is an algebraic variety, is contained in the union of no more than a countable set of such subvarieties B.

2. Consider a compact complex Kähler variety V of complex dimension n, on which there exists a nowhere-vanishing holomorphic n-form k. Such a form is, evidently, unique up to proportionality. Let $c_1, \dots, c_{b_n}$ be a basis of the free part of the homology group $H_n(V, \mathbf{Z})$. Put

$$\alpha_i = \int_{c_i} k.$$

These numbers, defined up to a common factor, may be regarded as projective coordinates of a certain point in the projective space P_{b_n-1}, where b_n is the n-th Betti number of the variety V.

Let $(\mathcal{V}, M, \pi)$ be a family of complex structures such that the base M is contractible, and $V_{t_0} = V$, where $t_0 \in M$. Suppose that on each variety V_t, where $t \in M$, there also exists a nowhere-vanishing holomorphic n-form k_t. Put

$$\alpha_i(t) = \int_{c_i} i_0^* p_t^* k_t,$$

where i_t^*, p_t^* are the homomorphisms of differential forms induced by the embedding $i_t : V_t \to \pi^{-1}(U)$ and the projection $p_t : \pi^{-1}(U) \to V_t$ ($\pi^{-1}(U)$ is diffeomorphic to $V_t \times U$).

Thus we obtain a smooth mapping

$$F : M \to P_{b_n-1}$$

of the base M into projective space.

This mapping can also be obtained in the following way. Choose, dual to the basis $c_1, \dots, c_{b_n}$, a basis $\omega_1, \dots, \omega_{b_n}$ of the space of n-dimensional harmonic forms on V_{t_0}. Since the form k_t, and consequently also the form $i_0^* p_t^* k_t$, is closed, we have the decomposition

$$i_0^* p_t^* k_t = \sum_{i=1}^{b_n} \alpha_i(t) \omega_i + d\eta(t).$$

The coefficients $\alpha_i(t)$ of this decomposition coincide with the homogeneous coordinates of the image of the point t under the mapping F.

Theorem 2. Let V be a compact complex Kähler variety of complex dimension n, on which there exists a nowhere-vanishing holomorphic n-form u, and, moreover, $H^{n-1}(V, \mathbf{C}) = 0$. Then the mapping F of the base M of an effectively parametrized family of deformations of complex structures of the variety V into the projective space $P_{b_n - 1}$ is holomorphic and is locally an embedding.

Remark 1. From the condition $H^{n-1}(V, \mathbf{C}) = 0$ it follows that n is even and all odd-dimensional cohomology groups $H^{2i+1}(V, \mathbf{C})$ are trivial.

Remark 2. The assertion of Theorem 2 is also valid in the case where the variety V is a complex torus.

3. In this section Theorems 1 and 2 are used to study deformations of structures on a compact Kähler surface V for which

$$q = H^1(V, \Omega^0) = 0, \qquad c_1(V) = 0,$$

where Ω^i is the sheaf of germs of holomorphic i-forms on the surface V, and $c_1(V)$ is the first Chern characteristic class of the surface V.

To this type of surface belongs a Kummer surface, as well as the surface considered in Kodaira’ s paper ([3]), on which there exist no meromorphic functions except constants.

It can be shown that the surface V has no torsion and that its cohomology groups have the form

$$H^1(V, \mathbf{Z}) = H^3(V, \mathbf{Z}) = 0,$$

$$\dim H^2(V, \mathbf{C}) = 22.$$

Indeed, it is not difficult to see that for the given surface

$$p = \dim H^0(V, \Omega^2) = 1.$$

The dimension $h^{1,1}$ of the space $H^1(V, \Omega^1)$ can be computed by Noether' s formula, which, as was observed in ([3]), is valid for an arbitrary compact Kähler surface. Consequently,

$$12(p - q + 1) = E + c_1^2(V),$$

where E is the Euler characteristic of the surface V. Taking into account that $E = 2 - 4q + 2p + h^{1,1}$, we obtain $h^{1,1} = 20$, whence $\dim H^2(V, \mathbf{C}) = 22$.

Let Θ be the sheaf of germs of complex-analytic vector fields on the surface V. By the duality theorem ([4]),

$$\dim H^q(V, \Theta) = \dim H^{n-q}(V, \Omega^1(K)).$$

Thus, for the surface under consideration,

$$H^0(V, \Theta) = H^2(V, \Theta) = 0,$$

$$\dim H^1(V, \Theta) = h^{1,1} = 20.$$

By virtue of the main theorem of ([5]), it follows from this that there exists a complete effectively parametrized family $(\mathcal{V}, M, \pi)$ of complex structures such that the base M is a 20-dimensional complex manifold and $V = V_{t_0} = \pi^{-1}(t_0)$, where $t_0 \in M$.

If U is a sufficiently small neighborhood of the point t_0 in the base M, then, according to Theorem 2, by means of the mapping F the neighborhood U can be identified with a certain neighborhood $F(U)$ on the 20-dimensional hypersurface K_{20} in the projective space P_{21}. This hypersurface is given by the equation

$$(z_1, \dots, z_{22})\, H\, (z_1, \dots, z_{22})' = 0,$$

where $z_1, \dots, z_{22}$ are homogeneous coordinates of the space P_{21}, and H is the intersection matrix of the surface V. Indeed, as we saw in § 2,

$$i_0^* p^* k_t = \sum_{i=1}^{22} \alpha_i(t) \omega_i + d\eta(t),$$

where $\alpha_i(t)$ are the projective coordinates of the point $F(t)$. Since $k_t \wedge k_t = 0$, we have

$$\sum_{i,j}\left(\int_V \omega_i \wedge \omega_j\right)\alpha_i(t)\alpha_j(t)=0.$$

Obviously, the matrix $H=\|h_{ij}\|$, such that $h_{ij}=\int_V \omega_i \wedge \omega_j$, coincides with the intersection matrix of the surface V.

Let C be a curve (a one-dimensional complex submanifold) on the surface V_t. Since

$$\int_C k_t = 0,$$

then

$$\beta_1\alpha_1(t)+\cdots+\beta_{22}\alpha_{22}(t)=0,$$

where the integers $\beta_1,\ldots,\beta_{22}$ are the coefficients in the expansion of the cycle C with respect to the homology basis $c_1,\ldots,c_{22}$. Thus, in order that there be at least one curve on the surface V_t, it is necessary that the point $F(t)$ lie on the intersection of the quadric K_{20} and the hyperplane $\sum \beta_i z_i = 0$, where the β_i are integers. Since on an algebraic surface there always lies at least one curve (a hyperplane section), it follows from this that there does not exist an effectively parametrized family of deformations of the structure of a surface V such that all the surfaces V_t are algebraic and the dimension of the base of this family is equal to 20. On the other hand, by Theorem 1, if V is an algebraic surface, then there exists an effectively parametrized family $(\mathcal{V}_A, M_A, \pi)$ of algebraic surfaces with base dimension 19 and such that $V=V_{t_0}$.

From the equality $H^1(V_t,\Omega')=0$ it follows that the group of divisor classes on the algebraic surface V_t is isomorphic to the group of integral 22-dimensional vectors $(\beta_1,\ldots,\beta_{22})$ such that

$$\sum \beta_i\alpha_i(t)=0,$$

where $\alpha_i(t)$ are the projective coordinates of the point $F(t)$. Hence it is clear that almost all (i.e., all except for the union of at most countably many subvarieties of smaller dimension) points of the base M_A of the family $(\mathcal{V}_A, M_A, \pi)$ correspond to algebraic surfaces on which all curves are multiples of a hyperplane section. Nevertheless, as is easy to show, the base number of such a surface (in the present case it coincides with the rank of the divisor class group) can take all possible values from 1 to 20.

The assertion that an algebraic surface of the type under consideration "has number of moduli" 19, as well as the last assertions about the base number, are contained in the works ([6],[7]). In Grauert' s survey ([8]) it is asserted that

Theorem 2 for the case of a Kummer surface was proved, but not published, by Andreotti.

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Note: Figure translations are in progress. See original paper for figures.