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Abstract

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ON ESTIMATING THE NORM OF A LINEAR OPERATOR IN A CLASS OF ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician V. I. Smirnov on May 6, 1963)

1. Let $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$ denote the class of entire functions $g(z_1, \dots, z_n)$ of degree $\leqslant (\nu_1, \dots, \nu_n)$ and satisfying the condition

$$(\|g\|_p^{(n)})^p = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |g(x_1, \dots, x_n)|^p \, dx_1 \cdots dx_n < +\infty.$$

Let, further, $\mathfrak{M}$ be the set of linear operators T possessing the following properties:

1°. T is defined on the set $W^{(p)}_{\nu_1,\dots,\nu_n}$, and its norm $\|T[g]\|_p^{(n)}$ is invariant with respect to any real shift in each argument.

2°. There exists a constant $A(\nu_1, \dots, \nu_n)$ such that

$$\|T[g]\|_\infty^{(n)} \leqslant A(\nu_1, \dots, \nu_n) \, \|g\|_p^{(n)},$$

where $\|f\|_\infty^{(n)}$ denotes the norm of the function $f(x_1, \dots, x_n)$ in the metric of the space $C^{(n)}(-\infty, \infty)$.

The first problem of the present note* is to establish, in the form of an inequality, a dependence between the various norms of different linear operators T and S from the set $\mathfrak{M}$ in the class of entire functions $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$. The solution of this problem is based on the fact that for any entire function $g(z) = g(x+iy)$ from the class $W^{(p)}_{\nu}$ $(p > 1)$, for any $z = x+iy$, the identity holds (see (1), p. 59)

$$g(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t - x) \, \frac{\sin \nu(t + iy)}{t + iy} \, dt. \tag{1}$$

By successive application of identity (1) to the function $g(z_1, \dots, z_n)$ with respect to each argument, we find that for an entire function $g(z_1, \dots, z_n)$ from the class $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$ we have:

$$g(x_1 + iy_1, \dots, x_n + iy_n) =$$

$$= \left(\frac{1}{\pi}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1 - x_1, \dots, t_n - x_n) \prod_{k=1}^{n} \frac{\sin \nu_k (t_k + iy_k)}{t_k + iy_k} \, dt_1 \cdots dt_n. \quad (2)$$

Hence, when $y_1 = y_2 = \cdots = y_n = 0$, it follows that for an entire function $g(z_1, \dots, z_n) \in W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$, for any real $x_1, \dots, x_n$, the identity holds

$$g(x_1, \dots, x_n) =$$

$$= \left(\frac{1}{\pi}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(t_1 - x_1, \dots, t_n - x_n) \prod_{k=1}^{n} \frac{\sin \nu_k t_k}{t_k} \, dt_1 \cdots dt_k. \quad (3)$$

* The results of the present note were reported at the International Congress of Mathematicians in August 1962 in Stockholm and at the Second All-Union Conference on Constructive Function Theory in October 1962 in Baku.

Moreover, we use the fact that if $g(z_1, \dots, z_n) \in W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$ and $\varphi(x_1, \dots, x_n) \in \mathcal{L}^{(n)}_{p'}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, then

$$F(z_1, \dots, z_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(z_1 + t_1, \dots, z_n + t_n) \varphi(t_1, \dots, t_n) \, dt_1 \cdots dt_n \quad (4)$$

is an entire function of the class $B_{\nu_1,\dots,\nu_n}$ (see (1), p. 38), and the function $\varphi(t_1, \dots, t_n)$ can be chosen so that the equality

$$F(0, 0, \dots, 0) = \|g\|^{(n)}_p \quad (5)$$

holds.

This leads to the following assertions:

Theorem 1. If $g(z_1, \dots, z_n)$ is an entire function of the class $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$; T, S are linear operators from the set $\mathfrak{M}$, then from the validity of the inequality*

$$\|T[g]\|^{(n)}_\infty \leqslant A(\nu_1, \dots, \nu_n) \|g\|^{(n)}_\infty$$

it follows that

$$\|T[g]\|_p^{(n)} \leqslant A(\nu_1, \dots, \nu_n)\|g\|_p^{(n)}$$

for every $p \geqslant 1$. Moreover, from the validity of the inequality

$$\|T[g]\|_\infty^{(n)} \leqslant \lambda(\nu_1, \dots, \nu_n)\|S[g]\|_\infty^{(n)}$$

it follows that

$$\|T[g]\|_p^{(n)} \leqslant \lambda(\nu_1, \dots, \nu_n)\|S[g]\|_p^{(n)}$$

for every $p \geqslant 1$.

In particular, if T is the differentiation operator, then from the validity of the classical inequality of S. N. Bernstein

$$\left|\frac{\partial g(x_1, \dots, x_n)}{\partial x_k}\right| \leqslant \nu_k \|g\|_\infty^{(n)}$$

there follows the Bernstein-Nikol' skii inequality (3)

$$\left\|\frac{\partial g(x_1, \dots, x_n)}{\partial x_k}\right\|_p^{(n)} \leqslant \nu_k \|g\|_p^{(n)}$$

for every $p \geqslant 1$.

Theorem 2. Let T, S be linear operators from the set $\mathfrak{M}$, and let $g(z_1, \dots, z_n)$ be an entire function of the class $W_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$. Then from the validity of the inequality

$$\|T[g]\|_\infty^{(n)} \leqslant \lambda(\nu_1, \dots, \nu_n)\|S[g]\|_\infty^{(n)}$$

it follows that

$$\|T[g]\|_\infty^{(n)} \leqslant \prod_{k=1}^{n} \left(\frac{\nu_k}{\pi}\right)^{1/p} \lambda(\nu_1, \dots, \nu_n)\|S[g]\|_p^{(n)}$$

for $1 \leqslant p \leqslant 2$, and, moreover,

$$\|T[g]\|_q^{(n)} \leqslant \left(\prod_{k=1}^{n} \frac{\nu_k}{\pi}\right)^{1/p-1/q} \lambda(\nu_1, \dots, \nu_n)\|S[g]\|_p^{(n)}$$

for $1 \leqslant p < q \leqslant +\infty$.

Theorem 3. If $g(z_1, \ldots, z_n)$ is an entire function of the class $W^{(p)}_{\nu_1,\ldots,\nu_n}$ $(p \geqslant 1)$, T is a linear operator from the set $\mathfrak{M}$, $1 \leqslant p \leqslant 2$,

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* Theorem 1 in the one-dimensional case was proved by another method in work (2).

$1 \leqslant p < q \leqslant +\infty$, then

$$\|T[g]\|^{(n)}_q \leqslant \left[\left(\frac{1}{\pi}\right)^n B_q^{n/q} \left(\prod_{k=1}^n \nu_k \right)^{1/p} \right]^{1-p/q} \|T[g]\|^{(n)}_p,$$

where

$$B_\infty = 1, \qquad B_q = \int_{-\infty}^{\infty} \left| \frac{\sin u}{u} \right|^q du.$$

Hence, in the case when T is the identity transformation, the inequality

$$\|g\|^{(n)}_q \leqslant \prod_{k=1}^n \left(\frac{s\nu_k}{\pi} \right)^{1/p-1/q} \|g\|^{(n)}_p$$

holds for all p and q satisfying the condition $1 \leqslant p < q \leqslant +\infty$, where $s = [[-p/2]]$ is the least integer not less than $p/2$. The last inequality is a refinement of S. M. Nikol' skii' s inequality (^{3}), which was generalized and sharpened in works ($^{4-6}$) for more general differential operators.

II. Let $\varphi(x_1, \ldots, x_n) \geqslant 1$ be a continuous function in n-dimensional Euclidean space (R_n), let $p \geqslant 1$ be any number, and let $\Lambda_{P,\varphi}$ be the class of functions $f(x_1, \ldots, x_n)$ possessing the property

$$\|f\|^{(n)}_{P,\varphi} = \left\| \ldots \left\{ \left\| \ldots \left(\left\| \frac{f}{\varphi} \right\|_{p_1} \right) \ldots \right\|_{p_k} \right\} \ldots \right\|_{p_n} < +\infty,$$

where $P = (p_1, p_2, \ldots, p_n)$ and $p_1, p_2, \ldots, p_n$ are various numbers not less than unity. In particular, for $n = 3$ the norm $\|f\|^{(3)}_{P,\varphi}$ has the form:

$$\|f\|^{(3)}_{P,\varphi} = \left\{ \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left| \frac{f(x_1, x_2, x_3)}{\varphi(x_1, x_2, x_3)} \right|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right]^{p_3/p_2} dx_3 \right\}^{1/p_3} < +\infty.$$

Obviously, the class $\Lambda^{(n)}_{P,\varphi}$, called the generalized Lebesgue class, coincides with the ordinary Lebesgue class $\mathcal{L}^{(n)}_p(-\infty,\infty)$ when

$$\varphi(x_1,\dots,x_n)\equiv 1, \qquad p_1=\dots=p_n=p.$$

Further, let $W^{(P,\varphi)}_{\nu_1,\dots,\nu_n}$ denote the class of entire functions $g(z_1,\dots,z_n)$ of finite degree $(\nu_1,\dots,\nu_n)$ that belong to the space $\Lambda^{(n)}_{P,\varphi}$. Obviously, the class $W^{(P,\varphi)}_{\nu_1,\dots,\nu_n}$ in the case $\varphi\equiv 1$ and $p_1=p_2=\dots=p_n=p$ coincides with the class $W^{(n)}_{\nu_1,\dots,\nu_n}$. In the case when $p_1,p_2,\dots,p_n$ are various numbers not less than unity, and $\varphi(x_1,\dots,x_n)\equiv 1$, the notations used are

$$W^{(p,1)}_{\nu_1,\dots,\nu_n}\equiv W^{(p_1,\dots,p_n)}_{\nu_1,\dots,\nu_n}, \qquad \|f\|^{(n)}_{P,1}=\|f\|_{p_1,\dots,p_n}.$$

The second problem* of the present note consists in establishing a connection between the different norms $\|g\|_{p_1,\dots,p_n}$ and $\|g\|_{p'_1,\dots,p'_n}$ of an entire function $g(z_1,\dots,z_n)$ from the class $W^{(p_1,\dots,p_n)}_{\nu_1,\dots,\nu_n}$, where $1\leqslant p_i<p'_i\leqslant\infty$ $(i=1,2,\dots,n)$.

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- A less precise result with respect to the constant was obtained by the author, by another method, in the work ([6]), carried out jointly with A. S. Dzhafarov, where a connection was established between the different norms $\|g\|^{(n)}_{P',\varphi}$ and $\|g\|^{(n)}_{P,\varphi}$ in the class $W^{(P,\varphi)}_{\nu_1,\dots,\nu_n}$, with $P=(p_1,\dots,p_n)$, $P'=(p'_1,\dots,p'_n)$, and $1\leqslant p_i<p'_i\leqslant\infty$ $(i=1,2,\dots,n)$.

1. If $g(z_1,\dots,z_n)\in W^{(p_1,\dots,p_n)}_{\nu_1,\dots,\nu_n}$ and $p_1,p_2,\dots,p_n$ are distinct numbers not less than one, then

$$\max_{-\infty<x_1,\dots,x_n<\infty}|g(x_1,\dots,x_n)|\le\prod_{k=1}^{n}\left(\frac{s_k\nu_k}{\pi}\right)^{1/p_k}\|g\|_{p_1,\dots,p_n},$$

where $s_k=|[-p_k/2]|$ is the least integer not less than $p_k/2$ $(k=1,2,\dots,n)$.

2. If $g(z_1,\dots,z_n)\in W^{(p_1,\dots,p_n)}_{\nu_1,\dots,\nu_n}$; $p_1,p_2,\dots,p_n$, $p'_1,p'_2,\dots,p'_n$ are distinct numbers not less than one, and $1\le p_i\le p'_i\le\infty$ $(i=1,2,\dots,n)$, then we have*

$$\|g\|_{p'_1,\dots,p'_n}\le\prod_{k=1}^{n}\left(\frac{s_k\nu_k}{\pi}\right)^{1/p_k-1/p'_k}\|g\|_{p_1,\dots,p_n}, \tag{7}$$

where

$$s_k = |[-p_k/2]| \quad (k = 1, 2, \dots, n).$$

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* A special case of inequality (7) with a nonsharp constant was considered independently of us by S. M. Nikol' skii ([7]).

Note: Figure translations are in progress. See original paper for figures.