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Abstract

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SOME GENERALIZATION OF THE SPACE OF K. FRIEDRICHS

(Presented by Academician S. L. Sobolev on 8 February 1963)

For the study of boundary-value problems of mathematical physics, variational methods are widely used. In the work of K. Friedrichs ([1]), in connection with the consideration of the quadratic functional

$$J(u) = (Au, u) - 2(u, f), \tag{1}$$

corresponding to the linear operator equation

$$Au = f \tag{2}$$

with a Hermitian positive-definite operator in a Hilbert space H, there was introduced a space with scalar product

$$[u, v]_A = (Au, v) \tag{3}$$

and with the corresponding norm

$$\|u\|_A = (Au, u)^{1/2}.$$

This space arises naturally in the solution of many concrete equations; however, in the general case it has the drawback that it is not complete. Therefore, as a rule, the space introduced by K. Friedrichs was completed in the usual way by ideal elements ([8]), and thereby the original operator A was in some sense extended.

Thanks to the introduction by S. L. Sobolev ([2]) of the concept of a generalized derivative and of the spaces W_p^l, the variational method for studying boundary-value problems for partial differential equations received a more rigorous and deeper interpretation. In this case the operator A in equation (2) is extended in advance so that the corresponding space immediately turns out to be complete. Further development of this direction was obtained in the works of V.

I. Kondrashov ([4]), S. M. Nikol' skii ([9]), L. D. Kudryavtsev ([5]), and other authors. By introducing a certain auxiliary operator, A. E. Martynyuk ([7]) and V. V. Petrishin ([10]) generalized the variational principle to a certain new class of equations. The spaces introduced in this connection, as also in ([1,8]), were completed by ideal elements.

In the present note a more general class of operators than in ([6,7,10]) is considered; a generalization of the concept of extension in the sense of K. Friedrichs is introduced for such operators, and also, by analogy with the generalized derivative, a generalization of the extension in the sense of S. L. Sobolev is introduced, and a certain relation between these extensions is established. Further, for the class of operators under consideration, a new space is introduced, which is a generalization of the space earlier proposed by K. Friedrichs ([1]). This space makes it possible to pose the variational problem for a sufficiently broad class of equations (2) with an operator A that is not necessarily self-adjoint and does not necessarily have a bounded inverse.

Let distributive operators A and B be given in a Hilbert space H. We shall denote the domains of definition of these operators respectively by $D(A)$, $D(B)$, their intersection $D(A)\cap D(B)$ by $D(A,B)$, the set $\{Au\}$ by $R(A)$, if the element u ranges over $D(A,B)$, and the set $\{Bu\}$

respectively through $R(B)$. We shall further assume that the sets $R(A)$ and $R(B)$ are dense in H and that the operator B admits closed extensions.

Definition 1. We shall call the operator A B**-symmetric** if, for any elements $u, v \in D(A,B)$, the equality

$$(Au, Bv) = (Bu, Av).$$

holds.

Operators close to these were considered in the works ([3], [6], [7]).

Definition 2. We shall call the operator A B**-positive** if the following relations hold: 1) $(Au, Bu) > 0$ for $u \neq 0$; 2) from the condition $(Au, Bu) \to 0$ it follows that $\|u\| \to 0$.

The class of B-symmetric operators is broader than the class of B-positive ones. It is not difficult to see that in a complex Hilbert space the B-symmetry of an operator follows immediately from the property of B-positivity. The B-positivity of an operator, introduced by Definition 2, is in a certain sense a necessary and sufficient condition for the solvability of the variational problem for the functional

$$D(u) = (Au, Bu) - 2(Bu, f),$$

corresponding to equation (2).

We now introduce the concept of extension of operators, which arises naturally in the solution of variational problems.

Definition 3. We shall say that, on elements $u, v \in H$, B**-extensions in the sense of K. Friedrichs** $\mathcal{A}$ and $\mathcal{B}$, corresponding to the operators A and B, are defined if in $D(A, B)$ there are sequences $\{u_k\}$ and $\{v_k\}$, strongly converging respectively to u and v, and elements ω_u, χ_v, such that the following relations hold: 1) $\lim\limits_{k\to\infty}(\omega_u - Au_k, \chi_v - Bv_k) = 0$; 2) $\lim\limits_{k\to\infty}[(\omega_u, \chi_v) - (Au_k, Bv_k)] = 0$. In this case we shall write $\omega_u = \mathcal{A}u,\ \chi_v = \mathcal{B}v$.

For the extension introduced above the following holds.

Theorem 1. *If the set $\{Bv\}$ is dense in H when v ranges over $D(B) \cap D(B^*A)$, then the B-extension in the sense of K. Friedrichs is unique in H.*

Moreover, the B-extension in the sense of K. Friedrichs has the following property:

Theorem 2. *The operator $\mathcal{A}$, corresponding to the B-symmetric B-positive operator A, will be $\mathcal{B}$-symmetric and $\mathcal{B}$-positive.*

If the operator A, in addition to the properties indicated above, also has the property that $D(A)$ is dense in H, then another concept of extension can be introduced:

Definition 4. We shall say that on an element $u \in H$ a B**-extension in the sense of S. L. Sobolev** $\widetilde{A}$, corresponding to the operator A, is defined if in H there is an element ω such that, for all $v \in D(A^*B) \cap D(B)$, the equality $(u, A^*Bv) = (\omega, Bv)$ holds. In this case we shall write $\omega = \widetilde{A}u$.

Analogously we shall also define the A-extension $\widetilde{B}$, corresponding to the operator B.

It is easy to see that if A is the differentiation operator, then, putting $B = I$ with domain of definition on smooth finite functions, we obtain the well-known definition of S. L. Sobolev ([2]) of the generalized derivative and, correspondingly, of the generalized differential operator.

Analogously to the results for the generalized derivative, the following holds.

Theorem 3. *In order that, on an element $u \in H$, the B-extension in the sense of S. L. Sobolev $\widetilde{A}$, corresponding to the operator A, be defined, it is sufficient that in $D(A)$ there be a sequence $\{u_k\}$ such that*

$$\lim_{k\to\infty}(u_k, A^*Bv) = (u, A^*Bv) \quad \text{for all } v \in D(A^*B), \qquad \|Au_k\| < c.$$

The proof of the theorem is carried out almost in the same way as in ([2]). With regard to the uniqueness of the extension introduced by Definition 4, the following is true.

Theorem 4. *The B-extension in the sense of S. L. Sobolev of the operator A is unique if the set $\{Bv\}$ is dense in H, as v ranges over $D(A^*B) \cap D(B)$.*

For many important cases, B-extensions in the sense of S. L. Sobolev have been well studied (see, for example, ([2],[4],[5],[9])), and therefore it is of interest to establish the relation between B-extensions in the sense of K. Friedrichs and in the sense of S. L. Sobolev.

Theorem 5. *Between the B-extensions in the sense of K. Friedrichs $\mathcal{A}$ and $\mathfrak{B}$ and the extensions in the sense of S. L. Sobolev $\widetilde{A}$ and $\widetilde{B}$, corresponding to the operators A and B, the relation holds*

$$D(A,B) \subseteq D(\mathcal{A},\mathfrak{B}) \subseteq D(\widetilde{A},\widetilde{B}).$$

This theorem gives, in many cases, the possibility of reducing extensions arising in the solution of variational problems to extensions in the sense of S. L. Sobolev, and of considering, for example, the spaces $W_2^{(l)}$ or the weighted spaces close to them ([5]).

With the aid of the $\mathfrak{B}$-positive operator $\mathcal{A}$, we now introduce the space $\mathfrak{F}_{A\mathfrak{B}}$, which we shall call the **K. Friedrichs space**. In the space $\mathfrak{F}_{A\mathfrak{B}}$ we define the scalar product in the following way:

$$[u,v]_{A\mathfrak{B}} = (\mathcal{A}u, \mathfrak{B}v)$$

and, accordingly, the norm

$$\|u\|_{A\mathfrak{B}} = (\mathcal{A}u, \mathfrak{B}u)^{1/2}.$$

The usual properties of the norm, as is not hard to verify, are fulfilled in this case.

Of the properties of the space $\mathfrak{F}_{A\mathfrak{B}}$, we note only the following:

Theorem 6. *The K. Friedrichs space $\mathfrak{F}_{A\mathfrak{B}}$ is complete.*

The introduced space $\mathfrak{F}_{A\mathfrak{B}}$ makes it possible to extend the ideas of S. L. Sobolev ([2]) to a significant class of equations with a non-self-adjoint operator.

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Note: Figure translations are in progress. See original paper for figures.