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Abstract

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DIFFERENTIAL EQUATIONS WITH MULTIVALUED DISCONTINUOUS RIGHT-HAND SIDE

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1°. Introduction.

The theory of differential equations with a multivalued right-hand side, developed by Zaremba and Marchaud in the 1930s (^{1,2}), has recently found applications to relay control systems, systems with hysteresis (^{3}), and to the theory of optimal regulation (^{4}). In (^{3,4}) equations $dx/dt \in F(t,x)$ are considered, where t is time, x is an n-dimensional vector, and $F(t,x)$ is a compact convex set which is a β-continuous function of (t,x), i.e. a function upper semicontinuous with respect to inclusion (^{3}). Such an equation is equivalent to an equation in contingencies (^{5}). The requirement of β-continuity of the function $F(t,x)$ with respect to (t,x) is in some cases too restrictive. Because of this, the theory of such equations does not include the theory of Carathéodory differential equations, which is needed for solving certain optimal regulation problems (see (^{6}), pp. 88-89). Problems in which all the given functions are infinitely differentiable may also lead to equations with a multivalued discontinuous right-hand side.

Suppose that in the system $dx/dt = f(t,x,u)$, $dy/dt = g(t,y,u,v)$ the functions f and g are infinitely differentiable. It is required to find a solution of some optimal problem for the first equation and, substituting the optimal control $u = u(t)$ found into the second equation, to find a solution of another optimal problem for this equation under the condition $v \in Q$, where Q is a given set. Thus it is necessary to find the optimal trajectory $y(t)$ among solutions of the equation $dy/dt \in R(t,y)$, where $R(t,y)$ is the set of values of the function $g(t,y,u(t),v)$ as v ranges over the set Q. If the function $u(t)$ is discontinuous on a set of positive measure (such a case is possible; see below, 2°), then the multivalued function $R(t,y)$ will not, generally speaking, be β-continuous in t, and the theory of equations in contingencies is not applicable.

In this note a class of differential equations with multivalued discontinuous right-hand side is singled out; it includes as special cases equations in contingencies, Carathéodory differential equations, and equations with single-valued discontinuous right-hand side considered in (^{7}). Solutions of such equations possess many of the basic properties of solutions of equations in contingencies.

2°. Example of a linear system in which the optimal control is discontinuous on a set of positive measure.

Suppose it is required to find a control $u(t)$, $|u(t)| \le 1$, for which the solution of the problem $dx/dt = b(t)u(t)$, $x(0) = 0$ assumes the greatest value at $t = 1$. Obviously, the solution is given by the formula $u(t) = \operatorname{sign} b(t)$ when $b(t) \ne 0$, while $u(t)$ is arbitrary when $b(t) = 0$. We construct the function $b(t)$ as follows. Take in the middle of the interval $[0, 1]$ an interval l_{11} of length $1/4$; in the middle of each of the remaining parts $[0, 3/8]$ and $[5/8, 1]$ take intervals l_{21} and l_{22} of length $1/4^2$ each. Continuing the construction indefinitely, we obtain a countable set of intervals l_{km}, $m = 1, 2, \dots, 2^{k-1}$; $k = 1, 2, 3, \dots$. On each interval l_{km} (its length is 4^{-k}) put

$$b(t) = (-1)^k 2^{-k^2} f(2^{2k+1}(t - a_{km})),$$

where a_{km} is the midpoint of the interval l_{km}, and

$$f(z) = e^{\frac{1}{z^2-1}}$$

for $|z| < 1$; on the perfect set of measure $1/2$ remaining

after removing these intervals, set $b(t) = 0$. The function $b(t)$ thus constructed for $0 \le t \le 1$ has discontinuous derivatives of all orders, while the optimal equation $u(t)$, equal to $(-1)^k$ on l_{km}, is discontinuous at all points of a set of measure $1/2$.

3°. The class of equations under consideration.

A. Let $F(t, x)$, where $x = (x_1, \dots, x_n)$, be a function defined in a domain D of the space $(t, x_1, \dots, x_n)$, or, at least, on almost all sections of this domain by planes $t = \text{const}$. For each point (t, x), $F(t, x)$ is a nonempty compact set in n-dimensional space $(v_1, \dots, v_n)$.

B. For each $t = \text{const}$, the function $F(t, x)$ is β-continuous (^{3}) with respect to x.

C. There exists a summable (on every finite interval) function $m(t)$ such that for all t, x we have $|v| \le m(t)$ for all $v \in F(t, x)$.

D. For any t, x, the set $F(t, x)$ is convex.

E. There exists a measurable single-valued vector function $f(t, x)$ such that $f(t, x) \in F(t, x)$ for all t, x.

A vector function $x(t)$, defined on an open or closed interval I, is called a solution (or trajectory) of the equation

$$\frac{dx}{dt} \in F(t, x), \tag{1}$$

if $x(t)$ is absolutely continuous and $dx(t)/dt \in F(t, x(t))$ for almost all $t \in I$.

Theorem 1. *Under conditions A—E, through any interior point of the domain D there passes at least one solution of equation (1). Every solution can be continued either indefinitely or up to the boundary of the domain D.*

4°. Measurable multivalued functions. Let, on a measurable set E of points $p = (p_1, \dots, p_m)$, there be defined a function $F(p)$; for any $p \in E$, $F(p)$ is a nonempty closed set in the space $R_n(v_1, \dots, v_n)$. The function $F(p)$ is called measurable if, for every closed set $D \subseteq R_n$, the set of those p for which the intersection of the sets $F(p)$ and D is nonempty is measurable. If, as D, one takes only closed (or open) balls with rational radii and coordinates of the center, then one obtains a definition equivalent to the present one; the definition from (^{8}) is also equivalent to the present one.

The multivalued function $F(p)$ is completely determined by specifying a countable set of single-valued functions $\rho_i(p) = \rho(a_i, F(p))$, where $a_1, a_2, \dots$ is a countable everywhere dense set in R_n, and ρ is distance. For measurability of $F(p)$, it is necessary and sufficient that all functions $\rho_i(p)$, $i = 1, 2, \dots$, be measurable; and for α-continuity (^{3}) (or β-continuity) of the bounded function $F(p)$, that all $\rho_i(p)$ be continuous (respectively, lower semicontinuous).

The proof is based on the fact that

$$\beta(F(p), F(q)) = \sup_i (\rho_i(q) - \rho_i(p)),$$

$$\alpha(F(p), F(q)) = \sup_i |\rho_i(q) - \rho_i(p)|.$$

Hence, and from Luzin' s theorem for single-valued functions $\rho_i(p)$, there follows a generalization of this theorem to multivalued functions: if $F(p)$ is almost everywhere bounded on a measurable set E, then, for its measurability, it is necessary and sufficient that, for any $\varepsilon > 0$, there exist a closed set E_ε such that the measure of the difference $E - E_\varepsilon$ is less than ε and on E_ε the function $F(p)$ is α-continuous. In (^{8}) this theorem is proved otherwise.

If the functions $F_k(p)$, $k = 1, 2, \dots$, are measurable, then their sum and intersection, upper and lower set-theoretic and topological limits, as well as the composite function $F(p) = f(p, F_1(p), \dots, F_s(p))$, are measurable, where $f(p, v^{(1)}, \dots, v^{(s)})$ is a single-valued continuous function of its arguments; $F(p)$ is the set traversed by the point $f(p, v^{(1)}, \dots, v^{(s)})$ when the $v^{(i)}$ independently of one another traverse $F_i(p)$. If $F(p)$ is measurable multivalued, then there exists a measurable single-valued $f(p) \in F(p)$.

5°. Properties of solutions.

Theorem 2. Let $x_k(t)$ be a solution of the equation

$$\frac{dx_k}{dt} \in F(t, x_k), \qquad k = 1, 2, \dots, \tag{2}$$

$$|dx_k(t)/dt| \leq m(t),$$

the function $m(t)$ is summable, the multivalued function $|F(t,x)|$ satisfies conditions A—Γ, item 3°,

$$\beta\left(F_k(t,x), \operatorname{konv} \sum_{\rho(x',x)<r_k(t)} F(t,x')\right) \leq \psi_k(t),$$

where $\operatorname{konv} \sum F$ is the smallest convex closed set containing the sum $\sum F$ of the sets F, $r_k(t) \to 0$, $\psi_k(t) \to 0$ (convergence in measure is sufficient). If $x_k(t) \to x(t)$, then $x(t)$ is a solution of equation (1).

Corollary 1. The limit of a convergent sequence of solutions of equation (1) (under conditions A—Γ, item 3°) is a solution of the same equation.

Corollary 2. If the solution $x(t)$ of equation (1) with initial conditions t_0, x_0 is unique and the conditions of Theorem 2 are fulfilled, then the solutions of equations (2) with initial conditions $x_k(t_k) = x_{k0}$ (where $t_k \to t_0,\ x_{k0} \to x_0$) converge to $x(t)$.

Corollary 3. If $F(t,x)$ satisfies conditions A—Γ, then the limit, as $k \to \infty$, of a convergent sequence of solutions of equations with delay

$$dx_k(t)/dt \in F(t, x_k(t - \tau_k)),$$

where $0 \leq \tau_k \leq 1/k$, is a solution of equation (1); here τ_k may depend on arbitrary factors, for example on t and on the values of $x_k(t)$ on the interval $[t_0, t]$.

The funnel of the point (t_0, x_0) is the set of points (t, x) lying on all solutions passing through this point. It is assumed that all these solutions for $t_0 \leq t \leq t_1$ lie strictly inside the domain D, in which equation (1) satisfies conditions A—D.

1) The segment $t_0 \leq t \leq t_1$ of the funnel is compact.

2) If H_k are the segments $t_0 \leq t \leq t_1$ of the funnels of the points (t_k, x_k), converging to the point (t_0, x_0), then $\beta(H_k, H) \to 0$, where H is the segment of the funnel of the point (t_0, x_0) for equation (1), and H_k is that for equation (1) or for equations (2) of Theorem 2.

3) The section of the funnel by the plane $t = \text{const}$ is a closed connected set.

4) Any point of the lateral boundary of the funnel can be joined to the point (t_0, x_0) by a segment of a trajectory passing entirely along the lateral boundary of the funnel. If, in addition, $F(t, x)$ is α-continuous in x for almost all t, then on this segment of the trajectory, almost everywhere dx/dt belongs to the boundary of the set $F(t, x)$.

5) If all solutions exist for $-\infty < t < \infty$, then in the space $(t, x_1, \dots, x_n)$ the solutions define a generalized dynamical system $(^{11,3})$.

6) The theorems of Lyapunov' s second method, in which the stability condition is the fulfillment of some inequality for the derivative φ' of a Lyapunov function taken along the system of differential equations, remain valid also for a system with a multivalued right-hand side, if

$$\partial\varphi/\partial t + v \operatorname{grad} \varphi$$

for all $v \in F(t, x)$ satisfies the same inequality as was required for φ'.

6°. Application to optimal control systems.

Consider the system (in vector notation)

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \qquad u(t) \in Q(t, x(t)), \tag{3}$$

where x and f are n-dimensional, u is an r-dimensional vector, the function f is measurable in (t, x, u), continuous in (x, u), $|f(t, x, u)| \leq m(t)$, $m(t)$ is summable; the set $Q(t, x)$ is a nonempty compact set in r-dimensional space; as a multivalued function $Q(t, x)$ is measurable in (t, x) and β-continuous in x. Let in the domain

for $t' \leq t \leq t''$ of the space (t, x) a bounded closed set A and a closed set B are given. It is required to find a solution of system (3) (an absolutely continuous function $x(t)$ and a measurable function $u(t)$ satisfying the system) such that the trajectory $x = x(t)$ passes from A to B (i.e., $(t_0, x(t_0)) \in A$, $(t_1, x(t_1)) \in B$, $t' \leq t_0 \leq t_1 \leq t''$), and such that the functional $\varphi(t_0, t_1, x(t))$, depending on the values of the solution $x(t)$ only on the interval $[t_0, t_1]$, assumes the smallest possible value.

Denote by $F(t, x)$ the set swept out by the vector $f(t, x, u)$ as u ranges over the set $Q(t, x)$. If $F(t, x)$ is convex for all t, x, then conditions A-D of item 3° are satisfied. Then the trajectories of system (3), for all admissible $u(t)$, coincide with the trajectories of equation (1) and, consequently, possess the properties listed in item 5°.

Suppose that there exists at least one admissible trajectory passing from A to B. Then, from the compactness of the set of trajectories issuing from points of the set A, it follows that there exists a trajectory $x(t)$ on which $\min \varphi(t_0, t_1, x(t))$ is attained, provided the functional is lower semicontinuous, i.e., if, when $t_0^k \to t_0$, $t_1^k \to t_1$, $x^k(t) \to x(t)$ (uniform convergence), $|dx^k/dt| \leq m(t)$, we have

$$\lim \varphi(t_0^k, t_1^k, x^k(t)) \geq \varphi(t_0, t_1, x(t)).$$

The magnitude of this minimum is a lower semicontinuous function of the data of the problem, i.e., of f, Q, A, B. These assertions remain valid if one regards as admissible only trajectories contained, for $t_0 \leq t \leq t_1$, in a given closed set D, as in (9), p. 30.

For the case when in (3) the function f is bounded (this can always be achieved by making in (3) the substitution of the independent variable t by τ, where $d\tau = (m(t)+1)\,dt$), we indicate one class of functionals that are lower semicontinuous. These are functionals of the form

$$\varphi_0(t_0, t_1, x(t_0), x(t_1), w_1, \dots, w_q, z_1, \dots, z_s),$$

where

$$w_j = \max_{t_0 \leq t \leq t_1} \psi_j(t, x(t)), \qquad z_i = \int_{t_0}^{t_1} g_i\left(t, x(t), \frac{dx(t)}{dt}\right) dt,$$

the functions φ_0 and ψ_j are continuous, φ_0 does not decrease in $z_1, \dots, z_s$, and the functions $g_i(t, x, v)$ are piecewise continuous in t (they may have discontinuities only of the first kind and only on a finite number of prescribed planes $t = \text{const}$), while $g_i(t, x, v)$ are convex downward in v, i.e., for $0 \leq \alpha \leq 1$

$$g_i(t, x, \alpha v_1 + (1-\alpha)v_2) \leq \alpha g_i(t, x, v_1) + (1-\alpha) g_i(t, x, v_2).$$

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Note: Figure translations are in progress. See original paper for figures.