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Abstract

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ON REPRESENTATIONS OF THE GROUP (p, p) OVER A FIELD OF CHARACTERISTIC p

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By a representation of a group G over a field K one means a homomorphism of the group G into the group of nonsingular matrices over the field K. In the case when the characteristic of the field K does not divide the order of the group, the theory of representations is well developed. The situation is much worse with representations of a group over a field whose characteristic divides the order of the group. If p is the characteristic of the field, then the problem of finding all representations of such groups reduces to the problem of finding all representations of p-groups. It is easy to find all representations of a cyclic group. V. A. Bashev (^{1}) found all representations of the group $(2, 2)$ over an algebraically closed field of characteristic 2. It would seem that the problem of finding all representations of the group (p, p) over a field of characteristic p, or at least of the group $(3, 3)$ over a field of characteristic 3, should be not much more difficult than the problem solved by V. A. Bashev. It turns out, however, that the problem of finding all representations over a field K of characteristic p of the group (p, p), for $p \neq 2$, is no easier than the problem of finding all representations of an arbitrary group over the field K. It is necessary to clarify the meaning of the words "to find all representations of the group G." In the case of a cyclic group and in the case considered by V. A. Bashev, the matrices of the representation are brought to a certain "normal" form. With the help of the normal form all indecomposable inequivalent representations are described.

In the present paper it is shown that if the problem of finding all representations of the group (p, p) is understood in the same sense and one aims to describe all representations of the group (p, p) over a field K of characteristic p by means of a normal form for the representation matrices, then for such a description it would be necessary to find (in the same sense) all representations of all groups over K.

1. Let G be an arbitrary group of order h; $g_1 = e, g_2, \dots, g_h$ its elements; φ a representation of G over a field K of characteristic p, $p \neq 2$; $\varphi(g_i) = X_i$, $i = 1, 2, \dots, h$; X_i of size $m \times m$; $\varphi(e) = X_1$ is equal to the identity matrix E. Two m-dimensional representations X_i and X_i', $i = 1, 2, \dots, h$, of the group G are called **equivalent** if there exists a matrix C such that $X_i' = C X_i C^{-1}$ for all i.

It is obvious that two representations X_i and X_i', $i = 1, 2, \dots, h$, are equivalent if and only if there exist nonsingular matrices P and Q such that $PX_iQ = X_i'$ for all i.

Indeed, $X_1 = X_1' = E$, and from $PX_1Q = X_1'$ we obtain that $P = Q^{-1}$. We shall assume that in the collection $X_1 = E, X_2, \dots, X_n$ there is an odd number of matrices, adding, if necessary, to the representation of the group G one more identity matrix.

Denote by $\overline{X}_i$ ($\widetilde{X}_i$) the matrix of size $m \times (m+1)$, the first m columns of which form exactly the matrix X_i (X_i'), while the $(m+1)$-st column is zero. Nonsingular matrices P and Q such that $PX_iQ = X_i'$, $i = 1, \dots, h$, exist if and only if there exist nonsingular matrices $\overline{P}$ and $\overline{Q}$ of sizes $m \times m$, $(m+1) \times (m+1)$, respectively, such that

$$\overline{P}\,\overline{X}_i\,\overline{Q} = \widetilde{X}_i', \quad i = 1, 2, \dots, h.$$

Indeed, if $PX_iQ = X_i'$, $i = 1, 2, \dots, h$, then as $\overline{P}$ and $\overline{Q}$ one may take the matrices

$$P, \quad \begin{pmatrix} Q & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \dots 0 & 1 \end{pmatrix}.$$

If, however, $\overline{P}\,\overline{X}_i\,\overline{Q} = \overline{X}_i'$, $i = 1, 2, \dots, h$, then

$$\begin{pmatrix} 0 \\ PX_i' \\ \vdots \\ 0 \end{pmatrix} \cdot \overline{Q} = \begin{pmatrix} 0 \\ X_i' \\ \vdots \\ 0 \end{pmatrix},$$

and since the matrix $\overline{P} \cdot \overline{X}_i$ is nonsingular, $\overline{Q}$ has the form

$$\overline{Q} = \left(\begin{array}{cc} \hat{Q} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * \dots * & * \end{array}\right),$$

where $\hat{Q}$ is a nonsingular matrix of size $m \times m$. As P and Q one may take $\overline{P}$ and $\hat{Q}$. Thus, the representations X_i and X_i', $i = 1, \dots, h$, are equivalent if and only if there exist nonsingular matrices $\overline{P}$ and $\overline{Q}$ of sizes $m \times m$, $(m+1) \times (m+1)$, respectively, such that

$$\overline{P}\,\overline{X}_i\,\overline{Q} = \overline{X}_i', \quad i = 1, 2, \dots, h.$$

2. $v = h - 1,\ v$ even. Let

$$X_i = (x^i_{kl}) \quad (k = 1, 2, \dots, m;\ l = 1, 2, \dots, m+1).$$

We introduce the matrices

$$\underbrace{\begin{pmatrix} x^1_{kl} & & & & \\ x^2_{kl} & x^1_{kl} & 0 & & \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \\ \vdots & x^2_{kl} & \vdots & & \\ x^h_{kl} & & \vdots & x^1_{kl} & \\ & x^h_{kl} & \vdots & x^2_{kl} & \\ & & \vdots & \vdots & \\ 0 & & & \vdots & x^h_{kl} \end{pmatrix}}_{mv} (m+1)v$$

$$\hat{S} = (S_{kl}) \quad (k = 1, 2, \dots, m;\ l = 1, 2, \dots, m+1).$$

By the letter S we shall denote the square nonsingular matrix

$$S = \begin{pmatrix} \hat{S} & E \\ E & 0 \end{pmatrix}, \qquad M_r = \underbrace{\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}_{r+1}}_{r}, \qquad N_r = \underbrace{\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{r+1}}_{r}.$$

For convenience we introduce the notation:

$$(m+1)v = \widetilde{m};$$

$$A_1 = \left(\begin{array}{ccc|c} M_{\widetilde{m}} & & & \\ & \ddots & & 0 \\ & & M_{\widetilde{m}} & \\ \hline & 0 & & M_{m\widetilde{m}} \end{array}\right), \qquad B_1 = \left(\begin{array}{ccc|c} N_{\widetilde{m}} & & & \\ & \ddots & & 0 \\ & & N_{\widetilde{m}} & \\ \hline & 0 & & N_{m\widetilde{m}} \end{array}\right),$$

$$A_2 = \left(\begin{array}{ccc|ccc} M_{mv-1} & & & & & \\ & \ddots & & & 0 & \\ & & M_{mv-1} & & & \\ \hline & & & M_1 & & \\ & 0 & & & \ddots & \\ & & & & & M_1 \end{array}\right), \qquad B_2 = \left(\begin{array}{ccc|ccc} N_{mv-1} & & & & & \\ & \ddots & & & 0 & \\ & & N_{mv-1} & & & \\ \hline & & & N_1 & & \\ & 0 & & & \ddots & \\ & & & & & N_1 \end{array}\right).$$

$$\underbrace{\qquad\qquad}_{\frac{m\widetilde{m}}{2}\ \text{times}} \qquad\qquad \underbrace{\qquad\qquad}_{\frac{m\widetilde{m}}{2}\ \text{times}}$$

$$A = \begin{pmatrix} E & A_1 & 0 \\ 0 & E & S \cdot A_2 \\ 0 & 0 & E \end{pmatrix}, \qquad A^p = \begin{pmatrix} E & p \cdot A_1 & \frac{p(p-1)}{2} A_1 \cdot S \cdot A_2 \\ 0 & E & p \cdot S \cdot A_2 \\ 0 & 0 & E \end{pmatrix} = E,$$

$$B = \begin{pmatrix} E & B_1 & 0 \\ 0 & E & S \cdot B_2 \\ 0 & 0 & E \end{pmatrix}, \qquad B^p = \begin{pmatrix} E & p \cdot B_1 & \frac{p(p-1)}{2} B_1 \cdot S \cdot B_2 \\ 0 & E & p \cdot S \cdot B_2 \\ 0 & 0 & E \end{pmatrix} = E,$$

since p is the characteristic of the field, which, by assumption, is not equal to 2,

$$AB = \begin{pmatrix} E & A_1 + B_1 & A_1 \cdot S \cdot B_2 \\ 0 & E & A_2 + B_2 \\ 0 & 0 & E \end{pmatrix}, \qquad BA = \begin{pmatrix} E & A_1 + B_1 & B_1 \cdot S \cdot A_2 \\ 0 & E & A_2 + B_2 \\ 0 & 0 & E \end{pmatrix}.$$

It is not difficult to verify that $AB = BA$. For this it suffices to prove that $A_1 S B_2 = B_1 S A_2$ (here the parity of v is essential).

3. Denote the generators of the group (p, p) by a and b. The defining relations of this group have the form $a^p = e,\ b^p = e,\ ab = ba$.

Since $A^p = E$, $B^p = E$ and $AB = BA$, the correspondence $a \mapsto A$, $b \mapsto B$ extends to a homomorphism of the group (p, p) into the group of nonsingular matrices over the field K of characteristic p. Consequently, the matrices A and B define a representation of the group (p, p). The matrix S' is constructed from the matrices $\bar{X}'_1, \bar{X}'_2, \dots, \bar{X}'_h$ in exactly the same way as S from $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_h$.

$$A' = \begin{pmatrix} E & A_1 & 0 \\ 0 & E & S' \cdot A_2 \\ 0 & 0 & E \end{pmatrix}, \qquad B' = \begin{pmatrix} E & B_1 & 0 \\ 0 & E & S' \cdot B_2 \\ 0 & 0 & E \end{pmatrix}.$$

A' and B' also define a representation of the group (p, p).

Let the representations A, B and A', B' be equivalent, i.e. there exists a matrix C such that $A' = CAC^{-1},\ B' = CBC^{-1}$. If we write C in block form: $C = (C_{ij})$ $(i, j = 1, 2, 3)$ and in the equalities $CA = A'C,\ CB = B'C$ multiply the matrices as block matrices, then we obtain a series of matrix equalities:

$$A_1C_{21} = 0, \quad B_1C_{21} = 0; \tag{1}$$

$$A_2C_{31} = 0, \quad B_2C_{31} = 0; \tag{2}$$

$$C_{21}A_1 = S'A_2C_{32}, \quad C_{21}B_1 = S'B_2C_{32}; \tag{3}$$

$$C_{11}A_1 = A_1C_{22}, \quad C_{11}B_1 = B_1C_{22}; \tag{4}$$

$$C_{22}SA_2 = S'A_2C_{33}, \quad C_{22}SB_2 = S'B_2C_{33}. \tag{5}$$

We shall regard the matrices A_1, B_1 as linear mappings of one and the same linear space of column vectors

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_\rho \end{pmatrix}, \qquad \rho = 2m(m+1)v, \quad \alpha_i \in K,$$

into another linear space. It is easy to establish that $\operatorname{Ker} A_1 \cap \operatorname{Ker} B_1 = 0$. In the analogous sense, $\operatorname{Ker} A_2 \cap \operatorname{Ker} B_2 = 0$. Therefore from equalities (1), (2), (3) we obtain that $C_{21} = 0,\ C_{31} = 0,\ C_{32} = 0$. Hence, and from the fact that C is a nonsingular matrix, it follows that the matrices C_{11}, C_{22}, C_{33} are nonsingular. Rename these matrices: $C_{11} = T,\ C_{22} = P,\ C_{33} = R$. Then equalities (4) and (5) take the form:

$$TA_1 = A_1P, \quad TB_1 = B_1P; \tag{4'}$$

$$PSA_2 = S'A_2R, \quad PSB_2 = S'B_2R. \tag{5'}$$

Denote the nonsingular matrix $S^{-1}PS$ by Q^{-1}. Then $S' = PSQ$.

Thus, we have proved that if the representation A, B of the group (p, p) is equivalent to the representation A', B', then $S' = PSQ$, where P and Q are nonsingular-

the given matrices satisfying the conditions:

$$TA_1 = A_1P, \qquad TB_1 = B_1P; \tag{4'}$$

$$QA_2 = A_2R, \qquad QB_2 = B_2R \tag{5''}$$

for some nonsingular matrices T and R.

$$T = \begin{pmatrix} T_{11} \; \dots \; T_{1m} & T_{1,m+1} \\ \vdots & \vdots \\ T_{m1} \; \dots \; T_{mm} & T_{m,m+1} \\ T_{m+1,1} \; \dots \; T_{m+1,m} & T_{m+1,m+1} \end{pmatrix} \begin{array}{l} \}\widetilde{m}+1 \\ \}\widetilde{m}+1 \\ \}m\widetilde{m}+1 \end{array} ,$$

$$P = \begin{pmatrix} P_{11} \; \dots \; P_{1m} & P_{1,m+1} \\ \vdots & \vdots \\ P_{m1} \; \dots \; P_{mm} & P_{m,m+1} \\ P_{m+1,1} \; \dots \; P_{m+1,m} & P_{m+1,m+1} \end{pmatrix} \begin{array}{l} \}\widetilde{m} \\ \}\widetilde{m} \\ \}m\widetilde{m} \end{array} .$$

From (4) we obtain:

$$T_{ij} M_{\widetilde{m}} = M_{\widetilde{m}} P_{ij}, \qquad T_{ij} N_{\widetilde{m}} = N_{\widetilde{m}} P_{ij} \quad (i, j = 1, 2, \dots, m),$$

$$T_{i,m+1} M_{m\widetilde{m}} = M_{\widetilde{m}} P_{i,m+1}, \qquad T_{i,m+1} N_{m\widetilde{m}} = N_{\widetilde{m}} P_{i,m+1} \quad (i = 1, 2, \dots, m).$$

From these matrix equalities it is easy to obtain that P_{ij} is a scalar matrix for $i, j = 1, 2, \dots, m$, and $P_{i,m+1} = 0$ for $i = 1, 2, \dots, m$. Consequently, P has the form:

$$P = \begin{pmatrix} c_{11}E & c_{12}E & \dots & c_{1m}E & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1}E & c_{m2}E & \dots & c_{mm}E & 0 \\ * & * & \dots & * & * \end{pmatrix} \begin{array}{l} \}\widetilde{m} \\ \}\widetilde{m} \\ \}m\widetilde{m} \end{array} .$$

Similarly, from conditions (5) we obtain that

$$Q = \begin{pmatrix} d_{11}E & d_{12}E & \dots & d_{1,m+1}E & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{m+1,1}E & d_{m+1,2}E & \dots & d_{m+1,m+1}E & * \\ 0 & 0 & \dots & 0 & * \end{pmatrix} \begin{array}{l} \}mv \\ \}mv \\ \}m\widetilde{m} \end{array} .$$

Introduce the notation:

$$\widehat{P} = \begin{pmatrix} c_{11}E & c_{12}E \dots c_{1m}E \\ c_{21}E & c_{22}E \dots c_{2m}E \\ \vdots & \vdots \\ c_{m1}E & c_{m2}E \dots c_{mm}E \end{pmatrix} \begin{array}{l} \}\widetilde{m} \\ \}\widetilde{m} \\ \}\widetilde{m} \end{array} , \qquad \widehat{Q} = \begin{pmatrix} d_{11}E & d_{12}E & \dots & d_{1,m+1}E \\ d_{21}E & d_{22}E & \dots & d_{2,m+1}E \\ \vdots & \vdots & & \\ d_{m+1,1}E & d_{m+1,2}E & \dots & d_{m+1,m+1}E \end{pmatrix} \begin{array}{l} \}mv \\ \}mv \\ \}mv \end{array} ,$$

$$\overline{P} = (c_{ij}), \qquad i, j = 1, 2, \dots, m; \qquad \overline{Q} = (d_{ij}), \qquad i, j = 1, 2, \dots, m+1.$$

For S', just as for S, the matrix $\widehat{S'}$ is defined. If $S' = PSQ$, then, obviously,

$$\widehat{S'} = \widehat{P}\,\widehat{S}\,\widehat{Q} \quad \text{and} \quad \overline{X}_i = \overline{P} X_i \overline{Q}, \quad i = 1, 2, \dots, h.$$

Thus, if the representations A, B and A', B' of the group (p, p) are equivalent, then the representations $X_1, X_2, \dots, X_h$ and $X_1', X_2', \dots, X_h'$ of the group

G are also equivalent. It is easy to see that if, conversely, the representations $X_1, \dots, X_h$ and $X'_1, \dots, X'_h$ of the group G are equivalent, then the corresponding representations A, B and A', B' of the group (p, p) are equivalent as well. The representation $X_1, \dots, X_h$ of the group G and the corresponding representation A, B of the group (p, p) are simultaneously decomposable or indecomposable.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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