ON THE QUESTION OF INEQUALITIES BETWEEN NORMS OF PARTIAL DERIVATIVES OF FUNCTIONS OF SEVERAL VARIABLES

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Abstract

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MATHEMATICS

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ON THE QUESTION OF INEQUALITIES BETWEEN NORMS OF PARTIAL DERIVATIVES OF FUNCTIONS OF SEVERAL VARIABLES

(Presented by Academician I. M. Vinogradov on 12 I 1963)

1. Let $f(x_1, \dots, x_n)$ be a continuous function, given in some domain D of the n-dimensional Euclidean space E^n of points $\mathbf{x} = (x_1, \dots, x_n)$, and having continuous derivatives of arbitrary order.

Let $n+1$ integer nonnegative vectors be given,

$$\mathbf{r}_i = (l_1^i, \dots, l_n^i)$$

($i = 0, 1, \dots, n;\ l_j^i \geq 0$ integers), with respect to which we require that

$$\begin{vmatrix} 1 & l_1^0 & \dots & l_n^0 \\ 1 & l_1^1 & \dots & l_n^1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & l_1^n & \dots & l_n^n \end{vmatrix} \neq 0. \tag{1}$$

Put

$$D^{\mathbf{r}} f = \frac{\partial^{l_1}}{\partial x_1^{l_1}} \cdots \frac{\partial^{l_n} f}{\partial x_n^{l_n}},$$

where $\mathbf{r} = (l_1, \dots, l_n)$, $l_j \geq 0$ are integers.

Suppose that

$$\|D^{\mathbf{r}_i} f\|_{L_p(D)} < \infty \qquad (i = 0, 1, \dots, n), \tag{2}$$

where $p \geq 1$.

The problem is to find the set of integer nonnegative vectors

$$\vec{\rho} = (v_1, \dots, v_n)$$

and the corresponding values of the parameter $q \geq p$ for which the inequality

$$\|D^{\vec{\rho}} f\|_{L_q(D)} \leq C \sum_{i=0}^{n} \|D^{\mathbf{r}_i} f\|_{L_p(D)} \tag{3}$$

will hold, where C is a constant independent of f.

Inequalities of the type of inequality (3) for the case when the vectors $\mathbf{r}_i$ are vectors of the form

$$\mathbf{r}_0 = (0, \dots, 0), \qquad \mathbf{r}_i = (0, \dots, l_i^i, \dots, 0) \quad (i = 1, \dots, n),$$

and the domain D coincides with the whole space E^n or is a rectangular parallelepiped with edges parallel to the coordinate axes, were first studied by S. M. Nikol' skii ([1])*; for $l_i^i = 1$ $(i = 1, \dots, n)$, results of the type of inequality (3) follow from still earlier works of S. L. Sobolev ([2]).

We shall agree to denote by Δ_R the rectangular parallelepiped in the space E^n of points $\mathbf{x} = (x_1, \dots, x_n)$, characterized by the inequalities

$$0 < x_i < R \qquad (i = 1, \dots, n).$$

In the present note necessary and sufficient conditions are formulated which must be imposed on the vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ and $\vec{\rho}$ in order that inequality (3) hold for $D = \Delta_R$ with some exponent $q > p$, and a number of results of negative character are also given.

2. We shall say that inequality (3), for given $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$, $\vec{\rho}$, p, q, and D, does not hold if, whatever constant C is chosen, there exists a function f, having in D continuous derivatives of arbitrary order and satisfy-

* S. M. Nikol' skii allowed not only integer but also fractional values l_i^i $(i = 1, \dots, n)$.

satisfying conditions (2), for which inequality (3) with this constant does not hold.

We shall write $\mathbf{r}_1 = (l_1^1, \dots, l_n^1) \geq \mathbf{r}_2 = (l_1^2, \dots, l_n^2)$ if $l_j^1 \geq l_j^2$ $(j = 1, \dots, n)$.

In what follows it will sometimes be more convenient for us to interpret the vectors $\mathbf{r}_i = (l_1^i, \dots, l_n^i)$ and $\vec{\rho} = (\nu_1, \dots, \nu_n)$ as points $M_i(l_1^i, \dots, l_n^i)$ and, respectively, $N(\nu_1, \dots, \nu_n)$ of the Euclidean space E^n. The set of points $M(l_1, \dots, l_n)$ of the space E^n whose coordinates satisfy the conditions $0 \leq l_i \leq \nu_i$ $(i = 1, \dots, n)$ will be denoted by $\delta(N)$.

Lemma 1. Suppose that integer nonnegative vectors $\mathbf{r}_i \geq 0$ $(i = 0, 1, \dots, n)$ and $\vec{\rho} \geq 0$ are given. If there exists a vector $\vec{\chi} = (\chi_1, \dots, \chi_n)$ (not necessarily integer, positive) such that

$$(\mathbf{r}_i, \vec{\chi}) = \sum_{j=1}^{n} l_j^i \chi_j = C_i \quad (i = 0, 1, \dots, n), \qquad (\vec{\rho}, \vec{\chi}) = \sum_{j=1}^{n} \nu_j \chi_j = C$$

and $C_i > C$ $(i = 0, 1, \dots, n)$ or $C_i < C$ $(i = 0, 1, \dots, n)$, then for $D = \Delta_R$ inequality (3) does not hold for any $q \geq p$.

Corollary. If the point N lies outside the n-dimensional simplex with vertices at M_i $(i = 0, 1, \dots, n)$, then for $D = \Delta_R$ inequality (3) does not hold for any $q \geq p$.

Lemma 2. Suppose one of the following conditions is fulfilled:

1) the points $M_i(l_1^i, \dots, l_n^i)$ $(i = 0, 1, \dots, n)$ are contained in $\delta(N)$;

2) the points M_i $(i = 0, 1, \dots, n)$ lie outside $\delta(N)$.

Then for $D = \Delta_R$ inequality (3) does not hold for any $q \geq p$.

Let now k points $M_0, \dots, M_{k-1}$, where $1 \leq k \leq n$, be contained in $\delta(N)$, and $n + 1 - k$ points $M_k, \dots, M_n$ be outside $\delta(N)$ (the point M_0, consequently, will always be considered a point of $\delta(N)$). Let

$$l_1\chi_1 + \dots + l_n\chi_n = C \geq 0 \tag{4}$$

be the equation of the hyperplane passing through the points $M_1, \dots, M_n$ (thus, it passes through all points exterior with respect to $\delta(N)$). Put

$$\nu_1\chi_1 + \dots + \nu_n\chi_n = C_1, \tag{5}$$

$$l_1^0\chi_1 + \dots + l_n^0\chi_n = C_2. \tag{6}$$

Lemma 3. Let $N(\nu_1, \dots, \nu_n)$ be a point of the simplex with vertices at M_i $(i = 0, 1, \dots, n)$, not belonging to the face (4) $(C_1 \neq C)$.

If among the coefficients χ_i of equation (4) there are nonpositive ones, for example, if $\chi_i > 0$ $(i = 1, \dots, k)$, $\chi_i = 0$ $(i = k + 1, \dots, m)$, $\chi_i < 0$ $(i = m + 1, \dots, n)$, then for $D = \Delta_R$ inequality (3) is possible only for $q = p$ and only in one of the following cases:

1) $C_2 < C_1 < C$, and the point M_0 lies in the hyperplane $l_{k+1} = \nu_{k+1}, \dots, l_n = \nu_n$;

2) $C_2 > C_1 > C$, and the point M_0 lies in the hyperplane $l_1 = \nu_1, \dots, l_m = \nu_m$.

Lemma 4. If the point $N(\nu_1, \dots, \nu_n)$ lies on the face of the simplex (with vertices at M_i $(i = 0, 1, \dots, n)$) containing the points $M_1, \dots, M_n$ $(C_1 = C)$, then, whatever the coefficients χ_i $(i = 1, \dots, n)$ of equation (4) may be, inequality (3) for $D = \Delta_R$ does not hold for $q > p$.

We note that the assertion analogous to this one, generally speaking, does not hold for points of other faces of the simplex under consideration.

Lemma 5. Suppose that for each point M_i $(i = k, \dots, n)$ exterior with respect to $\delta(N)$, at least one of the coordinates $l_1, \dots, l_s$, where $s < n$, is greater than the corresponding coordinate of the point $N(\nu_1, \dots, \nu_n)$.

Then inequality (3) for $D = \Delta_R$ can hold only for $q = p$ and only in the case when at least one of the points $M_0, \dots, M_{k-1}$ contained in $\delta(N)$ lies in the hyperplane $l_{s+1} = \nu_{s+1}, \dots, l_n = \nu_n$.

Corollary. From Lemma 5 it follows that inequality (3) for $q > p$ can hold only when in $\delta(IV)$ there is one point M_0, and outside $\delta(IV)$ there are n points $M_1, \dots, M_n$, with only one coordinate of each point M_i $(i = 1, \dots, n)$ greater than the corresponding coordinate of the point N, and for all these points these coordinates are different.

Theorem. In order that, for $D = \Delta_R$, inequality (3) hold for some $q > p$, it is necessary and sufficient that:

1) the coordinates of the vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ and $\vec{\rho}$ satisfy the inequalities:

$$l_j^0 \leqslant \nu_j \quad (j = 1, \dots, n),$$

$$l_j^i \leqslant \nu_j \quad (j = 1, \dots, n,\ j \neq i), \qquad l_i^i > \nu_i \quad \text{for } i = 1, \dots, n;$$

2) there exist numbers $\varkappa_j > 0$ $(j = 1, \dots, n)$ such that

$$l_1^i \varkappa_1 + \dots + l_n^i \varkappa_n = A \qquad (i = 1, \dots, n);$$

3)

$$\nu_1 \varkappa_1 + \dots + \nu_n \varkappa_n = A_1 < A.$$

If the indicated conditions are fulfilled, then there exists an interval of values of the parameter q for which (3) holds, determined from the relations

$$1 \leqslant p \leqslant q \leqslant \infty, \qquad A - A_1 - \left(\frac{1}{p} - \frac{1}{q}\right) \sum_{j=1}^{n} \varkappa_j = \varepsilon \geqslant 0,$$

where, if $\varepsilon = 0$, it is assumed that $1 < p < q < \infty$.

The necessity of conditions 1)–3) follows from Lemmas 5, 3, 4, and the sufficiency is proved by the method of integral representations.

Let us note that if the vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ satisfy the conditions of S. M. Nikol' skii, then condition 2) is fulfilled automatically, and for every point $N(\nu_1, \dots, \nu_n)$ of the simplex with vertices at M_i the inequalities 1) are valid.

3. Below, for the case $n = 2$, examples are given of various simplexes with vertices at M_i $(i = 0, 1, 2)$, possessing different properties in the sense of the question considered in the article.

In Fig. 1a a case is shown when inequality (3) for $D = \Delta_R$ does not hold for any vector $\vec{\rho}$ different from $\mathbf{r}_i$ $(i = 0, 1, 2)$. In Fig. 1 b, c, d, to the points N_i $(i = 1, 2, 3, 4)$ there correspond vectors $\vec{\rho}_i$ for which inequality (3) holds for $q = p$, while to the points of the triangles LNM_2 (Fig. 1c) and $M_0M_1M_2$ (Fig.

1d), except for the points of the segments LM_2 and M_1M_2, there correspond inequalities with $q > p$. These results are established by the method of integral representations. On the basis of the lemmas given above it is easy to establish that inequality (3) for $D = \Delta_R$ holds for no other vectors $\vec{\rho}$.

Fig. 1

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Note: Figure translations are in progress. See original paper for figures.