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Abstract

Full Text

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ON THE CONVERGENCE OF ORTHOGONAL LAGUERRE, HERMITE, AND JACOBI SERIES

(Presented by Academician G. I. Petrov on 18 IV 1963)

In the application of certain direct methods to differential equations (the Ritz, Galerkin, and least-squares methods), completeness of systems of approximating functions is required in spaces in which the norm of a function is defined by integrals of the squares of the function and its derivatives ⁽¹⁾. In the present paper, completeness in such spaces is proved for systems of Laguerre, Hermite, and Jacobi functions.

It is known that the orthonormal system of Laguerre functions

$$\varphi_{\alpha,n} = \frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}} e^{x/2} x^{-\alpha/2} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}) \quad (\alpha > -1, n = 0, 1, 2, \dots)$$

is complete in $L_2(0, \infty)$.

Denote by $W_2^{(k)}(0, \infty)$ the space whose elements are functions $\varphi(x)$, defined on the half-line $0 \leq x < \infty$, such that the derivatives up to order $k-1$ are absolutely continuous, the derivatives up to order k , together with $\varphi(x)$, belong to $L_2(0, \infty)$; at $x = 0$, $\varphi(x)$ and all its derivatives of order less than $\min(k, (\alpha-1)/2)$ are equal to zero, and the norm of the function $\varphi(x)$ in $W_2^{(k)}(0, \infty)$ is defined by the equality

$$\|\varphi\|^2 = \int_0^\infty \sum_{m=0}^k \left| \frac{d^m}{dx^m} \varphi(x) \right|^2 dx.$$

If $\alpha/2$ is not an integer, then we shall assume that $k < (\alpha+1)/2$, since derivatives of the Laguerre functions of higher order do not belong to $L_2(0, \infty)$; for integer $\alpha/2$, k is arbitrary.

Theorem 1. *The system of Laguerre functions is complete in $W_2^{(k)}(0, \infty)$.*

Consider the set of functions $\varphi(x)$ from $W_2^{(k)}(0, \infty)$ such that the functions $x^{-\alpha/2}\varphi(x)$ and their derivatives up to order $2k+1$ are absolutely continuous and

$$x^{\alpha/2+k+1} \frac{d^m}{dx^m} (x^{-\alpha/2} \varphi) \in L_2(0, \infty) \quad (m = 0, 1, 2, \dots, 2k + 2).$$

It is not difficult to show that this set is dense in $W_2^{(k)}(0, \infty)$; therefore it is sufficient to prove that the Fourier series of any function $\varphi(x)$ from this set with respect to the Laguerre functions converges strongly to $\varphi(x)$ in $W_2^{(k)}(0, \infty)$.

Let $\varphi(x)$ belong to the set under consideration. Its n -th Fourier coefficient is equal to

$$a_n = \int_0^\infty \varphi(x) \varphi_{\alpha, n}(x) dx.$$

Substituting here the explicit expression for $\varphi_{\alpha, n}$ and integrating by parts $(2k + 2)$ times for $n \geq 2k + 2$, we obtain

$$a_n = \frac{1}{\sqrt{n! \Gamma(n + \alpha + 1)}} \int_0^\infty \frac{d^{2k+2}}{dx^{2k+2}} (e^{x/2} x^{-\alpha/2} \varphi) \frac{d^{n-2k-2}}{dx^{n-2k-2}} (x^{n+\alpha} e^{-x}) dx$$

or

$$a_n = \sqrt{\frac{(n - 2k - 2)!}{n!}} b_{n-2k-2}, \quad (1)$$

where b_{n-2k-2} is the $(n - 2k - 2)$ -th Fourier coefficient of the function

$$x^{\alpha/2+k+1} e^{-x/2} \frac{d^{2k+2}}{dx^{2k+2}} (e^{x/2} x^{-\alpha/2} \varphi) \in L_2(0, \infty)$$

in its expansion in a series with respect to the functions $\varphi_{\alpha+2k+2, n}$.

It follows from (1) that the series

$$\sum_{n=0}^\infty n^{2k+2} |a_n|^2 \quad (2)$$

converges. For $\alpha > 1$, expand $\varphi'_{\alpha, n}$ in a Fourier series with respect to the functions $\varphi_{\alpha-2, m}$. We obtain

$$\varphi'_{\alpha, n} = \sum_{m=0}^n \left[n \left(1 - \frac{\alpha}{2} \right) + \frac{\alpha}{2} m + \frac{1}{2} \right] \sqrt{\frac{n! \Gamma(m + \alpha - 1)}{m! \Gamma(n + \alpha + 1)}} \varphi_{\alpha-2, m} + \frac{1}{2} \sqrt{\frac{n+1}{n+\alpha}} \varphi_{\alpha-2, n+1}. \quad (3)$$

For $\alpha = 0$, expand $\varphi'_{0,n}$ in a series with respect to the functions $\varphi_{0,m}$. We obtain

$$\varphi'_{0,n} = - \sum_{m=0}^n \varphi_{0,m} - \frac{1}{2} \varphi_{0,n}. \quad (4)$$

The coefficients in the expansion (3) have order $O(1)$ for $\alpha \geq 2$ and $O(\sqrt{n})$ for $\alpha < 2$; therefore,

$$\int_0^\infty (\varphi'_{\alpha,n})^2 dx = O(n) \quad \text{for } \alpha = 0 \text{ and } \alpha \geq 2,$$

$$\int_0^\infty (\varphi'_{\alpha,n})^2 dx = O(n^2) \quad \text{for } \alpha < 2.$$

Using (3) and (4), we obtain an estimate for derivatives of order higher than the first:

$$\int_0^\infty (\varphi_{\alpha,n}^{(m)})^2 dx = O(n^{2m}). \quad (5)$$

If $m < (\alpha - 1)/2$ or $\alpha/2$ is an integer, then in (5) one may replace $O(n^{2m})$ by $O(n^{2m-1})$.

We differentiate term by term m times the Fourier series of the function $\varphi(x)$. We shall show that the series

$$\sum_{n=0}^\infty a_n \varphi_{\alpha,n}^{(m)} \quad (6)$$

converges in the mean. Using the Cauchy–Bunyakovsky inequality and estimate (5), we obtain

$$\begin{aligned} \int_0^\infty \left| \sum_{n=p}^q a_n \varphi_{\alpha,n}^{(m)} \right|^2 dx &\leq \int_0^\infty \left(\sum_{n=p}^q n^{2k+2} |a_n|^2 \right) \sum_{n=p}^q \frac{(\varphi_{\alpha,n}^{(m)})^2}{n^{2k+2}} dx \\ &= \left(\sum_{n=p}^q n^{2k+2} |a_n|^2 \right) \sum_{n=p}^q \frac{O(n^{2m})}{n^{2k+2}}. \end{aligned}$$

From the convergence of series (2) and of the series

$$\sum_{n=1}^\infty \frac{O(n^{2m})}{n^{2k+2}} \quad (m \leq k)$$

there follows the convergence of series (6). To make sure that it converges to $\varphi^{(m)}(x)$, it is enough to verify the equality of the scalar products in $L_{2(0,\infty)}$ of the sum of the series and $\varphi^{(m)}(x)$ with any function from the complete system $x^{n+m}e^{-x/2}$ ($n = 0, 1, 2, \dots$). Thus, the Fourier series of the function $\varphi(x)$ converges strongly to $\varphi(x)$ in $W_{2(0,\infty)}^{(k)}$, and the theorem is proved.

Take the system, complete and orthogonal in $L_{2(-1,+1)}$,

$$\psi_{\alpha,\beta,n} = (1-x)^{\alpha/2}(1+x)^{\beta/2}P_n^{(\alpha,\beta)}(x)$$

$$(\alpha > -1, \beta > -1, n = 0, 1, 2, \dots),$$

where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. Denote by $W_{2(-1,+1)}^{(k)}$ the space whose elements are absolutely continuous functions $\psi(x)$ such that the derivatives up to order $(k-1)$ are absolutely continuous, the derivative of order k belongs to $L_{2(-1,+1)}$, $\psi(x)$ and its derivatives of order less than $\min(k, (\alpha-1)/2)$ are equal to zero at $x = +1$, $\psi(x)$ and its derivatives of order less than $\min(k, (\beta-1)/2)$ are equal to zero at $x = -1$, and the norm is defined by the equality

$$\|\psi\|^2 = \int_{-1}^{+1} \sum_{m=0}^k \left| \frac{d^m}{dx^m} \psi(x) \right|^2 dx.$$

If $\alpha/2$ is not an integer, then $k < (\alpha+1)/2$; the same applies to β .

Theorem 2. *The system of functions $\psi_{\alpha,\beta,n}$ ($n = 0, 1, 2, \dots$) is complete in $W_{2(-1,+1)}^{(k)}$.*

The proof is analogous to the proof of Theorem 1, although it requires more cumbersome calculations.

Denote by $W_{2(-\infty,+\infty)}^{(k)}$ the space whose elements are absolutely continuous functions $f(x)$, having absolutely continuous derivatives up to order $(k-1)$, with the derivatives up to order k , together with $f(x)$, belonging to $L_{2(-\infty,+\infty)}$, and the scalar product of functions $f(x) \in W_{2(-\infty,+\infty)}^{(k)}$ and $\varphi(x) \in W_{2(-\infty,+\infty)}^{(k)}$ is defined by the formula

$$(f, \varphi) = \int_{-\infty}^{+\infty} \sum_{m=0}^k \frac{d^m f}{dx^m} \frac{d^m \bar{\varphi}}{dx^m} dx. \quad (7)$$

Take the system of Hermite functions complete in $L_{2(-\infty,+\infty)}$

$$f_n = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \quad (n = 0, 1, 2, \dots).$$

Theorem 3. *The system of Hermite functions is complete in $W_{2(-\infty, +\infty)}^{(k)}$.*

The proof may be carried out analogously to the proof of Theorem 1. We shall give a shorter proof.

For the Hermite functions the following equality is known:

$$f_n = (-i)^n U f_n, \quad (8)$$

where U is the Fourier-Plancherel operator. Denote

$$P f = i \frac{df}{dx}.$$

From formula (8) it is easy to obtain

$$P f_n = (-i)^n U(x f_n), \quad M(P) f_n = (-i)^n U(M(x) f_n), \quad (9)$$

where $M(x)$ is an arbitrary polynomial. The system of functions $M(x) f_n$ ($n = 0, 1, 2, \dots$) is complete in $L_2(-\infty, +\infty)$, since the functions of this system are linear combinations of the functions of the complete system $M(x) e^{-x^2/4} x^n e^{-x^2/4}$ ($n = 0, 1, 2, \dots$), and conversely. The completeness of the latter system follows from the completeness of the system $x^n e^{-x^2/4}$ ($n = 0, 1, 2, \dots$) and the boundedness of the function $M(x) e^{-x^2/4}$. From (9) we obtain that the system $M(P) f_n$ ($n = 0, 1, 2, \dots$) is complete in $L_2(-\infty, +\infty)$ for an arbitrary polynomial $M(P)$. Let $f \in W_2^{(k)}(-\infty, +\infty)$ be orthogonal in $W_2^{(k)}(-\infty, +\infty)$ to all Hermite functions. Integrating by parts in formula (7), we obtain that f is orthogonal in $L_2(-\infty, +\infty)$ to the functions $M(P) f_n$

$$(n = 0, 1, 2, \dots), \quad \text{where } M(P) = \sum_{m=0}^k P^{2m},$$

and, consequently, f is equal to zero, which proves Theorem 3.

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REFERENCES

1. S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.

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