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Abstract

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MATHEMATICS

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ON THE DETERMINATION OF A STURM-LIOUVILLE DIFFERENTIAL EQUATION FROM TWO SPECTRA

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1. Consider the differential equation

$$y'' + \{\lambda - q(x)\}y = 0 \tag{1}$$

with boundary conditions

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(\pi) + Hy(\pi) = 0. \tag{3}$$

Here $q(x)$ is a real continuous function; h, H are real numbers.

Denote by $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n, \dots$ the eigenvalues of the problem (1) + (2) + (3), and by $\psi_0(x), \psi_1(x), \dots, \psi_n(x), \dots$ the corresponding eigenfunctions, normalized by the condition

$$\psi_n(0) = 1.$$

It is well known that if $q(x)$ is a sufficiently many times differentiable function, then, beginning with sufficiently large n, the asymptotic formulas

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + \cdots,$$

$$\alpha_n = \int_0^\pi \psi_n^2(x)\,dx = \frac{\pi}{2} + \frac{b_0}{n^2} + \frac{b_1}{n^4} + \cdots, \tag{4}$$

hold, where

$$a_0 = \frac{h + H + h_1}{\pi}, \qquad h_1 = \frac{1}{2}\int_0^\pi q(t)\,dt.$$

Replace condition (3) by the condition

$$y'(\pi) + H_1 y(\pi) = 0, \tag{3'}$$

where $H_1 \neq H$. Denote the eigenvalues of the problem (1) + (2) + (3′) by $\mu_0, \mu_1, \mu_2, \dots, \mu_n, \dots$.

By a known result of Borg ([1]), the numbers $\{\lambda_n\}$ and $\{\mu_n\}$ (where $n = 0, 1, 2, \dots$) uniquely determine the function $q(x)$, as well as the numbers h, H, and H_1.

The problem of effectively constructing the Sturm–Liouville equation from two spectra was studied by M. G. Krein, who obtained a number of fundamental results in this direction ([2,3]). In the present note another solution of this problem is given.

2. As I. M. Gel' fand and the author showed (see ([4]), especially § 11), equation (1) can be effectively reconstructed from the numbers $\{\lambda_n\}$ and $\{\alpha_n\}$, given together with their asymptotic expansions. Therefore the problem of the effective construction of equation (1) can be reduced to the computation of the num-

of the numbers α_n from the two spectra of equation (1). This can be done in the following way. Put*

$$\Phi_1(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \qquad \Phi_2(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\mu_n}\right).$$

Since $\lambda_n = O(n^2)$, $\mu_n = O(n^2)$, these infinite products converge for all λ and, consequently, are entire analytic functions.

It can be shown that

$$\alpha_k = \frac{\pi^2}{C_1 C_2 (H - H_1)}\,\Phi_1'(\lambda_k)\Phi_2(\lambda_k),$$

where

$$C_1 = \frac{1}{\lambda_0}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}, \qquad C_2 = \frac{1}{\mu_0}\prod_{n=1}^{\infty}\frac{n^2}{\mu_n}.$$

Theorem 1. Let

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right), \tag{5}$$

$$\sqrt{\mu_n} = n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + O\left(\frac{1}{n^4}\right), \tag{6}$$

with

$$a_0' \neq a_0. \tag{7}$$

Then

$$\alpha_k = \frac{\pi}{2} + \frac{b_0}{k^2} + O\left(\frac{1}{k^3}\right),$$

where

$$b_0 = \frac{\pi}{2}\left[-(\lambda_0+\mu_0) + 2(a_0+a_0') + \frac{a_1'-a_1}{a_0'-a_0} + a_0 + \frac{1}{6}\pi^2(a_0+a_0')^2 - (s_\lambda + s_\mu)\right],$$

with

$$s_\lambda = \sum_{n=1}^{\infty}(\lambda_n - n^2 - 2a_0), \qquad s_\mu = \sum_{n=1}^{\infty}(\mu_n - n^2 - 2a_0').$$

The proof is based on studying the asymptotic behavior of the expressions $\Phi_1'(\lambda_k)$ and $\Phi_2(\lambda_k)$ for large k. In principle, our method makes it possible to compute arbitrarily many terms of the asymptotic expansion (4), but even the determination of b_1 is associated with substantial computational difficulties. Therefore we restricted ourselves to computing b_0.

3. Our method also makes it possible to indicate necessary and sufficient conditions for two sequences of real numbers $\{\lambda_n\}$ and $\{\mu_n\}$ to be two spectra of a Sturm-Liouville equation.

We have proved the following theorem:

Theorem 2. Let the numbers $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the following conditions: a) the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ alternate; b) the asymptotic formulas (5) and (6) and condition (7) hold.

Then there exists an equation of the form (1) with a continuous function $q(x)$ and real numbers h, H, and H_1 such that the sequence $\{\lambda_n\}$ is the spectrum

* If $\lambda_j = 0$, then the factor $(1 - \lambda/\lambda_j)$ should be replaced by $-\lambda$.

of the problem (1) + (2) + (3), and the sequence $\{\mu_n\}$ is the spectrum of the problem (1) + (2) + (3′), and

$$a_0' - a_0 = \frac{1}{\pi}(H_1 - H).$$

If in the asymptotic expansions for $\sqrt{\lambda_n}$ and $\sqrt{\mu_n}$ there are k exact terms (not counting the first), then the function $q(x)$ is $(k-2)$ times continuously differentiable. In particular, in order that an infinite classical asymptotic expansion exist for $\sqrt{\lambda_n}$ and $\sqrt{\mu_n}$, it is necessary and sufficient that the function $q(x)$ be infinitely differentiable.

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REFERENCES

1. G. Borg, *Acta Math.*, **78**, No. 2, 1 (1945).
2. M. G. Krein, *Dokl. Akad. Nauk SSSR*, **76**, 21 (1951).
3. M. G. Krein, *Dokl. Akad. Nauk SSSR*, **76**, 345 (1951).
4. I. M. Gelfand, B. M. Levitan, *Izv. Akad. Nauk SSSR, Ser. Matem.*, **15**, 309 (1951).

Note: Figure translations are in progress. See original paper for figures.