Mathematics

V. F. Pisarenko

1963

SovietRxiv

Abstract

Full Text

Mathematics

V. F. Pisarenko

A Wiener–Hopf Type Equation for Multidimensional Random Processes with Rational Spectral Density

(Presented by Academician A. N. Kolmogorov, 22 XI 1962)

1. Let $\vec{\xi}(t) = (\xi_1(t), \dots, \xi_n(t))$ be an n-dimensional stationary random process in the broad sense with zero mean, whose spectral function is absolutely continuous, and whose spectral-density matrix $f(\lambda)$ has the form:

   $$f(\lambda) = (f_{kj}(\lambda))_{k,j=1,\dots,n} = (2\pi)^{-1}(Q_{kj}(z)P_{kj}(z^*)M_{kj}^{-1}(z)N_{kj}^{-1}(z^*))_{k,j=1,\dots,n},$$

   where $Q_{kj}(z),\ P_{kj}(z),\ M_{kj}(z),\ N_{kj}(z)$ are polynomials with real coefficients. For continuous (discrete) time $z = i\lambda$ ($z = e^{i\lambda}$), and the roots of the listed polynomials lie in the left half-plane (outside the unit circle), while the polynomials $Q_{kj}(z), P_{kj}(z)$ may also have purely imaginary roots (roots on the unit circle). Let $L^2(f)$ be the Hilbert space of complex-valued vector functions $\vec{\varphi}(\lambda) = (\varphi_1(\lambda), \dots, \varphi_n(\lambda))$ with scalar product:

   $$(\vec{\varphi}, \vec{\psi}) = \int \left[\sum_{k,j=1}^{n} f_{kj}(\lambda)\varphi_j(\lambda)\psi_k^*(\lambda)\right] d\lambda, \tag{1}$$

   and $L_T^2(f)$ the subspace of $L^2(f)$ generated by all functions of the form $C\vec{\psi}(\lambda)$, where C is a complex matrix and $\vec{\psi}(\lambda) = (e^{i\lambda t_1}, \dots, e^{i\lambda t_n})$, with $0 \le t_k \le T,\ k = 1, \dots, n$. By $\mathcal{L}_T(m)$ we shall denote the class of scalar functions whose $(m-1)$-st derivative is continuous and whose m-th derivative is square-integrable on the interval $[0, T]$. The degrees of the polynomials $M_{kk}(z)$ and $Q_{kk}(z)$ will be denoted by r_k and s_k, respectively. Consider a Wiener–Hopf type equation on a finite time interval:

   $$\int e^{-i\lambda t} f(\lambda)\vec{\varphi}(\lambda)\, d\lambda = \mathbf{u}(t), \qquad 0 \le t \le T, \tag{2}$$

   where $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))$ is a given function, and $\vec{\varphi}(\lambda)$ is the unknown function, $\vec{\varphi}(\lambda) \in L_T^2(f)$. For continuous time the integration in (1), (2) is performed from $-\infty$ to $+\infty$; for discrete time, from $-\pi$ to $+\pi$.

Many statistical problems for stationary processes reduce to solving equation (2) (see (1-4)). In the present note we generalize the results of work (1) (see also

(2)) for solving the one-dimensional equation (2) to the multidimensional case with a matrix of rational spectral densities (m.r.s.d.) $f(\lambda)$ that is nondegenerate almost everywhere (a.e.).

2. We shall call a nondegenerate a.e. m.r.s.d. $f(\lambda)$ strongly nondegenerate if the matrix

$$\lim_{\lambda\to\infty}\left((-i\lambda)^{r_k-s_k}f_{kj}(\lambda)(i\lambda)^{r_j-s_j}\right)_{k,j=1,\dots,n}$$

is nondegenerate. Lemmas 1 and 2 allow the solution of (2) for a nondegenerate a.e. m.r.s.d. $f(\lambda)$ to be reduced to the case of a strongly nondegenerate $f(\lambda)$.

Lemma 1. *For every nondegenerate a.e. m.r.s.d. $f(\lambda)$ one can find a matrix $H(i\lambda)$ possessing the following properties: a) the elements $H(i\lambda)$ are polynomials in $i\lambda$ with real coefficien-*

m.r.s.d.; b) the determinant $H(i\lambda)$ is equal to a nonzero constant; c) the matrix $f_1(\lambda) = H^*(i\lambda)f(\lambda)H(i\lambda)$ is a strongly nondegenerate m.r.s.d. ($H^*(i\lambda)$ is the matrix adjoint to $H(i\lambda)$).

Lemma 2. Let $f(\lambda)$ satisfy the conditions of Lemma 1, and let $f_1(\lambda)$ be the matrix mentioned in that lemma. Let $\vec{\varphi}(\lambda)$ be a solution of equation (2). Then there exists a function $\mathbf{g}(t) = H'(d/dt)\mathbf{u}(t)$, $0 < t < T$, and the function $\vec{\psi}(\lambda) = H^{-1}(i\lambda)\vec{\varphi}(\lambda)$ belongs to $L_T^2(f_1)$ and is a solution of the equation

$$\int_{-\infty}^{\infty} e^{-i\lambda t} f_1(\lambda)\vec{\psi}(\lambda)\,d\lambda = \mathbf{g}(t), \qquad 0 \le t \le T.$$

Theorem 1. Let $f(\lambda)$ be a strongly nondegenerate m.r.s.d. Then the space $L_T^2(f)$ consists of functions $\vec{\varphi}(\lambda) = (\varphi_1(\lambda), \dots, \varphi_n(\lambda))$, whose components have the form:

$$\varphi_k(\lambda) = R_1^{(k)}(\lambda) + R_2^{(k)}(\lambda)\int_0^T e^{i\lambda t}c_k(t)\,dt, \qquad k = 1, \dots, n,$$

and only of such functions (up to functions equal to zero a.e.). Here $R_1^{(k)}, R_2^{(k)}$ are certain polynomials of degree not exceeding $(r_k - s_k - 1)$ and $(r_k - s_k)$, respectively; $c_k(t)$ are square-integrable functions.

Theorem 2. Let $f(\lambda)$ satisfy the conditions of Lemma 1, and let $H(i\lambda)$, $f_1(\lambda)$ be the matrices mentioned in that lemma. In order that there exist in $L_T^2(f)$ a solution of equation (2), it is necessary and sufficient that the function $H'(d/dt)\mathbf{u}(t) = \mathbf{g}(t)$ exist and that its k-th component $g_k(t)$ belong to $\mathcal{L}_T(\rho_k - \sigma_k)$ ($2\rho_k$ and $2\sigma_k$ are the degrees of the polynomials in the denominator and numerator of the k-th diagonal element of the matrix $f_1(\lambda)$). If

$f(\lambda)$ is strongly nondegenerate, then the necessary and sufficient condition is $u_k(t) \in \mathcal{L}_T(r_k - s_k)$, $k = 1, \dots, n$. The solution is always unique.

In order not to encumber the exposition, assume that, for any j, $1 \leq j \leq n$, the polynomials $M_{kj}(i\lambda)$, $k = 1, \dots, n$, have no common factors, and likewise $N_{kj}(i\lambda)$, $k = 1, \dots, n$. Introduce the polynomials $M_j(i\lambda)$ and $N_j(i\lambda)$, as well as the polynomial matrix $P(i\lambda)$:

$$\prod_{k=1}^{n} M_{kj}(i\lambda) = M_j(i\lambda) = \sum_{k=0}^{\alpha_j} m_k^{(j)} (i\lambda)^k; \qquad \prod_{k=1}^{n} N_{kj}(i\lambda) = N_j(i\lambda) = \sum_{k=0}^{\beta_j} n_k^{(j)} (i\lambda)^k;$$

$$j = 1, \dots, n; \tag{3}$$

$$P(i\lambda) = \left(Q_{kj}(i\lambda)\, P_{kj}(-i\lambda)\, M_{kj}^{-1}(i\lambda)\, N_{kj}^{-1}(-i\lambda)\, M_j(i\lambda)\, N_j(-i\lambda)\right)_{k,j=1,\dots,n}.$$

Let $\mathbf{u}(t)$ be such that $u_k(t) \in \mathcal{L}_T(2r_k - 2s_k)$. Then the solution of equation (2), $\vec{\varphi}(\lambda)$, has the form

$$\varphi_k(\lambda) = \sum_{j=0}^{r_k - s_k - 1} (a_j^{(k)} + e^{i\lambda T} b_j^{(k)})(i\lambda)^j + \int_0^T e^{i\lambda t} M_k(-d/dt) N_k(d/dt) x_k(t)\, dt,$$

$$k = 1, \dots, n, \tag{4}$$

$$a_j^{(k)} = \sum_{m=j+\mu_k+1}^{\beta_k} \sum_{i=0}^{\alpha_k} n_m^{(k)} m_i^{(k)} (-1)^{i+j} x_k^{(m+i-j-1)}(+0);$$

$$b_j^{(k)} = -\sum_{m=0}^{\beta_k} \sum_{i=j+\nu_k+1}^{\alpha_k} n_m^{(k)} m_i^{(k)} (-1)^{i+j} x_k^{(m+i-j-1)}(T-0);$$

$$\mu_k = \beta_k - r_k + s_k, \qquad \nu_k = \alpha_k - r_k + s_k$$

and $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ is a solution of the system of equations

$$P(-d/dt)\mathbf{x}(t) = \mathbf{u}(t), \qquad 0 < t < T, \tag{5}$$

satisfying the boundary conditions:

$$M_k(-d/dt)x_k^{(i)}(+0) = 0, \quad i = 0, \dots, \mu_k - 1; \qquad N_k(d/dt)x_k^{(i)}(T-0) = 0,$$
$$i = 0, \dots, \nu_k - 1; \qquad k = 1, \dots, n. \tag{6}$$

Conditions (6) uniquely determine the solution of system (5). The general solution of system (5) is written down without difficulty (see (5,6)).

Let us now consider the case when $u_k(t) \in \mathcal{L}_T(r_k - s_k)$, $k = 1, \dots, n$. First suppose that the matrix $f(\lambda)$ is diagonal and $Q_{kk}(i\lambda) \equiv P_{kk}(i\lambda) \equiv 1$. Consider the function $g(t)$, for which

$$g_k(t) = \int_0^t (t-\tau)^{r_k - 1} u_k(\tau)\, d\tau.$$

Obviously, $g_k(t) \in \mathcal{L}_T(2r_k)$. Therefore the solution of equation (2) (where $g(t)$ stands instead of $\mathbf{u}(t)$), denoted by $\vec{\psi}(\lambda)$, can be found by formula (4). Let $\vec{\psi}(\lambda)$ have the form (4). The desired solution $\vec{\varphi}(\lambda)$ is obtained by substituting $\vec{\psi}(\lambda)$ into (2) and then differentiating the resulting identity. In order to write $\vec{\varphi}(\lambda)$ explicitly, we express the higher derivatives of the diagonal terms $R_{kk}(t)$ of the correlation function $R(t)$ in terms of the lower ones, using the differential equations for $R_{kk}(t)$ (see (7), p. 489):

$$R_{kk}^{(r_k+j)}(t) = \sum_{m=0}^{r_k-1} v_{jm}^{(k)} R_{kk}^{(m)}(t), \quad t > 0; \qquad R_{kk}^{(r_k+j)}(t) = \sum_{m=0}^{r_k-1} w_{jm}^{(k)} R_{kk}^{(m)}(t), \quad t < 0;$$

$$j = 0, \dots, r_k - 1; \qquad k = 1, \dots, n.$$

Then the solution $\vec{\varphi}(\lambda)$ has the form:

$$\varphi_k(\lambda) = \sum_{j,m=0}^{r_k-1} (-i\lambda)^m (-1)^j \left(a_j^{(k)} v_{jm}^{(k)} + e^{i\lambda T} b_j^{(k)} W_{jm}^{(k)}\right) +$$
$$+ (i\lambda)^{r_k} \int_0^T e^{i\lambda t} M_k(-d/dt) N_k(d/dt) x_k(t)\, dt.$$

The case of arbitrary $f(\lambda)$ is reduced to the preceding one. Let $\mathbf{x}(t)$ be a solution of system (5) satisfying conditions (6). Consider the system

$$\int_{-\infty}^{\infty} e^{-i\lambda t} M_k^{-1}(i\lambda) N_k^{-1}(-i\lambda) \varphi_k(\lambda)\, d\lambda = 2\pi x_k(t), \quad 0 \le t \le T, \quad k = 1, \dots, n. \tag{7}$$

Since $x_k(t) \in \mathcal{L}_T(\alpha_k + \beta_k - r_k + s_k)$ and system (7) is diagonal, its solution can be found by the method described above. It can be shown that the resulting $\vec{\varphi}(\lambda)$ belongs to $L_2^n(f)$ and is the desired solution of equation (2).

3. Let us now consider an autoregression process. In this case

$$f(\lambda) = (2\pi)^{-1} D^{-1}(z)[D^{-1}(z^*)]', \quad z = i\lambda, \quad \text{where } D(z) = \sum_{k=0}^{r} d_k z_k \tag{8}$$

is a matrix of polynomials in z with real coefficients, and all roots of $\det D(z)$ lie in the left half-plane (d_k are real numerical matrices). Suppose that $D(-i\lambda)$ and $D'(i\lambda)$ are permutable for all λ, and that the function $\mathbf{u}(t)$ has all derivatives occurring in formulas (9)–(10). In this case the solution of equation (2)

can be written at once in final form:

$$\vec{\varphi}(\lambda) = \sum_{k=0}^{r-1} \left(\mathbf{a}_k + e^{i\lambda T}\mathbf{b}_k\right)(i\lambda)^k + \int_0^T e^{i\lambda t} D'(d/dt) D(-d/dt)\mathbf{u}(t)\, dt; \tag{9}$$

$$\mathbf{a}_k = \sum_{m=k+1}^{r} \sum_{j=0}^{r} d'_m d_j (-1)^{j+k} \mathbf{u}^{(m+j-k-1)}(+0); \tag{10}$$

$$\mathbf{b}_k = -\sum_{m=0}^{r} \sum_{j=k+1}^{r} d_j d'_m (-1)^{j+k} \mathbf{u}^{(m+j-k-1)}(T-0).$$

4. Processes with discrete time. In this case equation (2) can be written in terms of the correlation function $R(t)$:

$$\sum_{\tau=0}^{T} R(t-\tau)\vec{\Phi}(\tau) = \mathbf{u}(t), \qquad 0 \leqslant t \leqslant T, \tag{11}$$

where $\vec{\Phi}(\tau)$ is the Fourier transform of the function $\vec{\varphi}(\lambda)$. It can be shown that for a nonsingular p.d. $f(\lambda)$ the solution of equations (2), (11) always exists and is unique. Introduce the symbol Δ for a time shift: $\Delta\Phi(t) = \Phi(t+1)$; $\Delta^{-1}\Phi(t) = \Phi(t-1)$. We shall use the notation of item 2. Let

$$T \geqslant \max_j \alpha_j + \max_j \beta_j.$$

Then the solution of (11) has the form

$$\Phi_k(t) = \sum_j \sum_m m_j^{(k)} n_m^{(k)} h_k(t+m-j), \qquad k = 1, \ldots, n, \quad 0 \leqslant t \leqslant T,$$

where the summation is over the following ranges: $0 \leqslant j \leqslant t$, $0 \leqslant m \leqslant \beta_k$ for $0 \leqslant t \leqslant \alpha_k - 1$; $0 \leqslant j \leqslant \alpha_k$, $0 \leqslant m \leqslant \beta_k$ for $\alpha_k \leqslant t \leqslant T - \beta_k$; $0 \leqslant j \leqslant \alpha_k$, $0 \leqslant m \leqslant T - t$ for $T - \beta_k + 1 \leqslant t \leqslant T$. The function $\mathbf{h}(t) = (h_1(t) \dots h_n(t))$ is the solution of the system of difference equations

$$P(\Delta)\mathbf{h}(t) = \mathbf{u}(t), \qquad 0 \leqslant t \leqslant T, \tag{12}$$

satisfying the boundary conditions:

$$M_k(\Delta^{-1})h_k(t) = 0, \qquad T < t \leqslant T + \gamma_k; \qquad N_k(\Delta)h_k(t) = 0, \qquad -\delta_k \leqslant t < 0;$$

$$k = 1, \dots, n, \tag{13}$$

where

$$\delta_j = \max_k(\alpha_j - \operatorname{st} M_{kj} + \operatorname{st} Q_{ki}), \qquad \gamma_j = \max_k(\beta_j - \operatorname{st} N_{kj} + \operatorname{st} P_{ki}).$$

The operator $P(\Delta)$ is obtained from formula (3) by replacing $-i\lambda$ by Δ and $+i\lambda$ by Δ^{-1}. Conditions (13) uniquely determine the solution of system (12). For an autoregression process, when $f(\lambda)$ has the form (8), where $z = e^{i\lambda}$, and when the matrices $D(z)$ and $D'(z^*)$ commute, the roots of $\det D(z)$ have moduli greater than 1 and $T \geqslant 2r$, one can immediately write down the solution of equation (11):

$$\vec{\Phi}(t) = \sum_m \sum_k d_m d'_k \, \mathbf{u}(t + k - m), \qquad 0 \leqslant t \leqslant T,$$

where the summation is over the following ranges: $0 \leqslant m \leqslant t$, $0 \leqslant k \leqslant r$ for $0 \leqslant t \leqslant r - 1$; $0 \leqslant m \leqslant r$, $0 \leqslant k \leqslant r$ for $r \leqslant t \leqslant T - r$; $0 \leqslant m \leqslant r$, $0 \leqslant k \leqslant T - t$ for $T - r + 1 \leqslant t \leqslant T$ (in this case the matrices d_m and d'_k must be interchanged).

The author expresses his sincere gratitude to R. L. Dobrushin for valuable comments.

Moscow State University
named after M. V. Lomonosov

Received
18 XI 1962

REFERENCES

1. V. F. Pisarenko, Yu. A. Rozanov, *Problems of Information Transmission*, vol. 14, 1963.

2. J. Hájek, *Czechoslovak Mathematical Journal*, **12** (87), 404 (1962).

3. A. M. Yaglom, *Theory of Probability and Its Applications*, **5**, No. 3, 265 (1960).

4. V. S. Pugachev, *Theory of Random Functions and Its Application*, Moscow, 1957.

5. F. R. Gantmacher, *Theory of Matrices and Its Applications*, Moscow, 1950.

6. J. Sansone, *Ordinary Differential Equations*, Moscow, 1954.

7. J. Doob, *Stochastic Processes*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.