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Abstract

Full Text

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ON CONDITIONS FOR UNIQUENESS OF A CONTINUOUS SOLUTION OF THE RIEMANN PROBLEM FOR A SYSTEM OF THREE EQUATIONS

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Consider the hyperbolic system of three quasilinear equations

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial s} = 0, \tag{1}$$

where $u = \{u^{(1)}, u^{(2)}, u^{(3)}\}$, $f = \{f^{(1)}, f^{(2)}, f^{(3)}\}$. Hyperbolicity of the system means that the eigenvalues λ_i of the matrix $\partial f/\partial u$ are all real. We shall assume that they are distinct and that the corresponding eigenvectors l_i are linearly independent,

$$\lambda_1 < \lambda_2 < \lambda_3; \qquad |\, l_1, l_2, l_3 \,| \neq 0. \tag{2}$$

The Riemann problem consists (see ([1])) in finding a solution of the system (1), in general discontinuous, satisfying discontinuous initial data: for $t = 0$, $u = u^-$ for $s < 0$, $u = u^+$ for $s > 0$, where u^- and u^+ do not depend on s. We shall restrict ourselves to considering only continuous (for $t > 0$) solutions depending on s/t. When the number of equations in the system (1) is less than three, such a solution is unique. Below an example is constructed of a system of three equations having a nonunique solution, and conditions for uniqueness are given.

Putting $s/t = \lambda$, we obtain from (1)

$$\lambda \frac{du}{d\lambda} = \frac{df(u)}{d\lambda} \tag{3}$$

or

$$\frac{du}{d\lambda} = l_i, \qquad \lambda = \lambda_i. \tag{3'}$$

The curves $u_i(\lambda)$ in the space u satisfying $(3')$ are called characteristic curves of the i-th family; denote them by L_i. A solution in the space u is represented by a sequence of three (or fewer) arcs L_1, L_2, L_3, which together give the transition from u^- to u^+. A continuous solution is unique if such a transition is unique.

As is known (see ([1], [2])), a necessary, and sometimes sufficient, condition for uniqueness of a generalized solution is the convexity condition

$$(l_i \cdot \nabla \lambda_i) \neq 0, \tag{4}$$

meaning that λ_i varies monotonically along L_i.

We shall now construct a system for which, in some domain of the space u, conditions (2), (4) are satisfied and nevertheless there is no uniqueness. Make a change of variables: from $u^{(1)}, u^{(2)}, u^{(3)}$ pass to x, y, z, defined by the equations

$$(l_2 \cdot \nabla x) = 0, \qquad (l_2 \cdot \nabla y) = 0, \qquad (l_2 \cdot \nabla z) = 1, \tag{5}$$

i.e., we take for x, y a pair of independent Riemann invariants for L_2, and $z = \lambda_2$. Putting $\varphi = zu - f$, by virtue of (3) and (5) we obtain $\varphi_z = u$, i.e., system (1) can be written in the form

$$\frac{\partial \varphi_z}{\partial t} + \frac{\partial}{\partial s}(z\varphi_z - \varphi) = 0, \tag{6}$$

and the characteristic system (3) in the form

$$d\varphi + \delta\, d\varphi_z = \varphi_z\, dz, \tag{7}$$

where $\delta = \lambda - z$. In the new variables l and L we denote, respectively, by m and M. M_2 are straight lines parallel to the z-axis, to which $\delta = \delta_2 = 0$ corresponds. M_1 and M_3 are determined from (7) for $\delta = \delta_1 < 0$ and $\delta = \delta_3 > 0$.

For simplicity let us require that M_1 and M_3 lie in the planes $z = \text{const}$; then (7) can be written in the form

$$d'\varphi + \delta\, d'\varphi_z = 0, \tag{8}$$

where d' contains differentiation only with respect to x and y. We need to construct three functions $\varphi(x, y, z)$ satisfying (8) for $\delta = \delta_1$ and $\delta = \delta_3$. We take one of them, $\varphi^{(3)} = z^2/2$, which obviously satisfies (8) for any δ. From the remaining, as yet undetermined, $\varphi^{(1)}(x, y, z)$, $\varphi^{(2)}(x, y, z)$, we form the matrix

$$K = \begin{pmatrix} \varphi_x^{(1)} & \varphi_y^{(1)} \\ \varphi_x^{(2)} & \varphi_y^{(2)} \end{pmatrix} \tag{9}$$

and rewrite (8) in the form

$$(K + \delta K_z)m' = 0, \tag{10}$$

where m' is the two-dimensional eigenvector of the first or the third family.

We shall obtain nonuniqueness of the solution if M_1, taken at $z = z_1$, and M_3, taken at $z = z_3 > z_1$, do not form a regular coordinate net when superposed in the plane x, y. But in this case there must exist a point x_0, y_0 (say $0, 0$) at which the curves of the resulting net are tangent.

The matrix

$$K^0 = \begin{pmatrix} 1+\cos z & 1+\sin z \\ -1+\sin z & 1-\cos z \end{pmatrix} \tag{11}$$

realizes this possibility for $z_1 = 0,\ z_3 = \frac{3}{2}\pi$. Indeed, by direct substitution we verify that for

$$\delta_{1,3} = \mp 1; \qquad m'_{1,3} = \{-\sin\tfrac{1}{2}(z - z_{1,3}),\ \cos\tfrac{1}{2}(z - z_{1,3})\} \tag{12}$$

the relation (10) is identically fulfilled, while the vectors $m'_1(z_1)$ and $m'_3(z_3)$ are equal to $\{0, 1\}$, i.e., they coincide. We have defined K on the straight line $x = 0,\ y = 0$. It remains to define it in a neighborhood of this line so that the convexity condition (4) be satisfied; it is now written in the form

$$(m' \cdot \nabla\delta) \neq 0. \tag{13}$$

Differentiate (10) with respect to x; we obtain

$$(K_x + \delta K_{zx})m' + \delta_x K'_{zm} + (K + \delta K_z)m'_x = 0. \tag{14}$$

We define m^* by the condition

$$(K^* + \delta K^*_z)m^* = 0. \tag{15}$$

Multiplying (14) scalarly by m^*, we obtain

$$((K_x + \delta K_{zx})m' \cdot m^*) + \delta_x(K'_{zm} \cdot m^*) = 0. \tag{16}$$

Differentiating (10) with respect to y, we obtain an analogous equality, a linear combination of which with (16) gives

$$(m' \cdot \nabla\delta) = -\frac{m^{(1)}((K_x + \delta K_{zx})m' \cdot m^*) + m^{(2)}((K_y + \delta K_{zy})m' \cdot m^*)}{(K'_{zm} \cdot m^*)}. \tag{17}$$

In order that $(m' \cdot \nabla\delta)$ not change sign for $x = 0,\ y = 0,\ z_1 \leq z \leq z_3$, it is necessary that the numerator and denominator of the expression on the right-hand side of (17) not vanish. The denominator has already been determined, and since

$$m^*_{1,3} = \{-\sin \tfrac{1}{2}(z + z_{3,1}),\ \cos \tfrac{1}{2}(z + z_{3,1})\}, \tag{18}$$

we easily obtain that

$$(K_z^0 m' \cdot m^*)_{1,3} = \mp \frac{1}{\sqrt{2}}.$$

From the definition of the matrix K (9) it follows that the second column of K_x must coincide with the first column of K_y; otherwise they are arbitrary. Setting

$$K_x^0 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K_y^0 = q \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where $q = \frac{1}{2}(e^{-z} - e^{z-z_3})$, and using (12), (18), (19), we obtain

$$\tfrac{1}{2}(m' \cdot \nabla\delta)_{1,3} = \pm \sin^3\left(\frac{z - z_{1,3}}{2}\right) + e^{\mp z - z_{1,3}} \cos^3\left(\frac{z - z_{1,3}}{2}\right).$$

It is not difficult to verify that the last expression is different from zero for $0 = z_1 \leq z \leq z_3 = \frac{3}{2}\pi$.

Thus, we have constructed the matrix

$$K = K^0 + xK_x^0 + yK_y^0$$

and, consequently, a pair of functions $\varphi^{(1)}, \varphi^{(2)}$, which, together with $\varphi^{(3)} = z^2/2$, generate a system of the form (6) having, in a neighborhood of the segment $x = 0,\ y = 0,\ 0 \leq z \leq \frac{3}{2}\pi$, a nonunique solution. In this case conditions (2), (4) are satisfied, and the Jacobian of the transformation from u to x, y, z, equal to $|K_z|$, is different from zero.

The most essential point in the constructed example is the realization of the case $m_1'(z_1) = m_3'(z_3)$. We shall now show that the absence of the effect $m_1' = m_3'$ ensures uniqueness of the solution of the posed problem.

Suppose that one can pass from u^- to u^+ in two ways. Then on the portion not coinciding in these two transitions one of the following three cases occurs: a) each transition consists entirely of one arc L; b) one of them consists of two arcs L of different families; c) both consist of two arcs. Consider case a): the points u^- and u^+ are joined, for example, both by an arc L_1 and by an arc L_3. We pass to the variables x, y, z defined above. Projecting M_1 and M_3 onto the plane $z = \text{const}$, we obtain M_1' and M_3'. The closed curve M_1' and M_3' bounds a domain Σ', which is the projection of the surface Σ spanned by the arcs M_1 and M_3. On $\Sigma + M_1 + M_3$ a continuous vector field m_1 is defined, which generates a vector field m_1' on $\Sigma' + M_1' + M_3'$. The latter is continuous everywhere except at the self-intersection points $M_1 + M_3$, and has no singular points, since m_1 and m_2 are not collinear. Let us first suppose that $M_1' + M_3'$ has no self-intersections. Under the displacement

along $M_1' + M_3'$ the field vector m_1' cannot change its orientation (from internal to external or vice versa), since otherwise at the point where it changes m_1, m_2, m_3 would prove to be coplanar. Consequently, in going around $M_1' + M_3'$ the vector m_1' turns through a nonzero angle, and this is impossible, since in Σ' there are no singular points of the field m_1', i.e. $M_1' + M_3'$ must have a point of self-intersection (let $x = 0$, $y = 0$). This point is the projection of a segment of the z-axis, which is a characteristic curve of the second family M_2, and we obtain case b). Consider one of the parts Σ' into which it is divided by the self-intersection of $M_1' + M_3'$, with vertex at the point $0, 0$. We shall denote this part also by Σ'. Remove from it a small neighborhood ε' of the point $0, 0$. Then either for $\Sigma' - \varepsilon'$ the case already considered applies, or on the portion of the boundary ε' lying in Σ' there occurs a change in the orientation of the field m_1' relative to M_3'. But displacement along this portion of the boundary ε' is equivalent to displacement along the segment of the z-axis projected to $0, 0$. Therefore on this segment there will be found a point where m_1' is tangent to M_3'. Consequently, on the z-axis there are two points giving the effect $m_1' = m_3'$. It is not difficult to show that case c) leads to the same conclusion. The presence or absence of the effect $m_1' = m_3'$ can be related to the possibility or impossibility of constructing, for each point x, y, a pair of vectors separating m_1' and m_3' for all z. If such vectors exist, then the vector fields generated by them may be taken as coordinate fields, i.e. the coordinates x, y can be chosen so that the vectors m_1' and m_3' will always lie in different octants.

Thus, for uniqueness of the continuous solution of the problem posed, it is necessary and sufficient that among the solutions of the equation

$$(l_2 \cdot \nabla\psi) = 0$$

there be found two linearly independent solutions ψ_1, ψ_2 such that

$$(l_1 \cdot \nabla\psi_1)(l_1 \cdot \nabla\psi_2)(l_3 \cdot \nabla\psi_1)(l_3 \cdot \nabla\psi_2) < 0.$$

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[2] P. D. Lax, *Comm. Pure and Appl. Math.*, **10**, No. 4, 537 (1957).

Note: Figure translations are in progress. See original paper for figures.