HYDROMECHANICS

Corresponding Member of the Academy of Sciences of the USSR L. N. SRETENSKII

1963

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

Corresponding Member of the Academy of Sciences of the USSR L. N. SRETENSKII

PERIODIC WAVES CREATED BY A SOURCE LOCATED ABOVE A SLOPING BOTTOM

The present article contains the results of a study of waves arising from a source with periodically varying discharge and located at a given depth above a uniformly descending bottom. The entire solution of the problem is carried out under the assumptions of plane-parallel potential motions; at the same time it is assumed that the angle α of inclination of the bottom to the horizon is an integral fraction of 90°, i.e. $\alpha = \pi/2n$.

Denote by $w(z,t)$ the characteristic function of the flow and put

$$w(z,t) = f(z)\cos\sigma t,$$

where σ is the given frequency of variation of the source discharge. The function $f(z)$ must satisfy the conditions:

$$\operatorname{Im}\left(\frac{df}{dz} + i\nu f\right) = 0 \quad \text{for } z = x; \qquad \operatorname{Im}\left(e^{i\alpha}\frac{df}{dz}\right) = 0 \quad \text{for } \arg z = -\alpha,$$

the parameter ν has the following value: $\nu = \sigma^2/g$.

In addition, we require the function $f(z)$ to be holomorphic near the point $z = 0$, where the free surface of the liquid intersects the bottom of the basin.

The source of oscillations is located at the point $z = \rho e^{-\mu i}$; near this point the function $f(z)$ will have the form

$$f(z) = \frac{q}{2\pi}\ln\left(z - \rho e^{-\mu i}\right) + \cdots$$

To determine the function $f(z)$ one uses the differential equation (1)

$$\sum_{k=0}^{n-1} a_k\left[f^{(k+1)}(z) + i\nu f^{(k)}(z)\right] = \chi(z) + h(z), \tag{1}$$

in which the constant coefficients a_k are determined by the formula

$$a_0 = 1, \quad a_k = \frac{1}{\nu^k} \operatorname{ctg} \alpha \operatorname{ctg} 2\alpha \ldots \operatorname{ctg} k\alpha \quad (k = 1, 2, \ldots, n-1).$$

The functions $\chi(z)$ and $h(z)$ have the following expressions:

$$\chi(z) = \frac{i\nu q}{2\pi} \ln \left\{ \frac{(z-c_0)(z-c_1')(z-c_2)(z-c_3') \ldots (z-c_{2n-1}')}{(z-c_0')(c-c_1)(z-c_2')(z-c_3) \ldots (z-c_{2n-1})} \right\},$$

$$h(z) = \frac{q}{2\pi} \sum_{k=0}^{n-1} (-)^k k! a_k \sum_{j=0}^{2n-1} (-)^j \left[\frac{1}{\left(ze^{-2\alpha ij} - \rho e^{-\mu i}\right)^{k+1}} + \frac{1}{\left(ze^{-2\alpha ij} - \rho e^{\mu i}\right)^{k+1}} \right] -$$

$$-\frac{i\nu q}{2\pi} \sum_{k=1}^{n-1} (-)^k (k-1)! a_k \sum_{j=0}^{2n-1} (-)^j \left[\frac{1}{\left(ze^{-2\alpha ij} - \rho e^{-\mu i}\right)^{k}} - \frac{1}{\left(ze^{-2\alpha ij} - \rho e^{\mu i}\right)^{k}} \right],$$

where

$$c_s = \rho e^{-\mu i + 2\alpha i s}, \qquad c_s' = \rho e^{\mu i + 2\alpha i s}.$$

To integrate the equation with constant coefficients (1), we note that the roots of the equation

$$\sum_{k=0}^{n-1} a_k \left(\lambda^{k+1} + i\nu\lambda^k\right) = 0$$

are

$$\lambda_0 = -i\nu, \quad \lambda_1 = -i\nu\varkappa, \quad \lambda_2 = -i\nu\varkappa^2, \ldots, \lambda_{n-1} = -i\nu\varkappa^{n-1},$$

where $\varkappa = e^{-2\alpha i}$.

The integral of equation (1), satisfying the above-mentioned conditions: the wave condition and the condition of flow around the basin bottom, is written as follows:

$$f(z) = \sum_{p=0}^{n-1} i^p \left\{ ae^{-\frac{1}{4}\pi i(n-1)} + \frac{(-)^p \varkappa^p \nu^{n-1}}{\Lambda'(\lambda_0)} \int_0^z [\chi(\xi) + h(\xi)] e^{-\lambda_p \xi} \, d\xi \right\} e^{\lambda_p z} \operatorname{ctg} \alpha \operatorname{ctg} 2\alpha \ldots \operatorname{ctg} p\alpha.$$

Here a is an arbitrary real number and

$$\Lambda'(\lambda_0) = (2\nu)^{n-1} e^{-\frac{1}{4}\pi i(n-1)} \sin\alpha \, \sin 2\alpha \ldots \sin(n-1)\alpha.$$

The values of the function $f(z)$ for large real values $z = x$ are determined by the formula

$$f(x) = \{[A + a\cos\tfrac{1}{4}\pi(n-1)] + i[B - a\sin\tfrac{1}{4}\pi(n-1)]\}(\cos\nu x - i\sin\nu x),$$

where

$$A + Bi = \frac{\nu^{n-2}}{\Lambda'(\lambda_0)} \int_0^\infty \left[\nu h(\xi) - \frac{q\nu}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\xi - c_j} - \frac{1}{\xi - c'_j}\right)\right] e^{-\lambda_0\xi} d\xi.$$

Performing integration by parts, we obtain another expression for $A + Bi$:

$$A + Bi = \frac{q}{2\pi}\frac{\nu^{n-1}}{\Lambda'(\lambda_0)}\sum_{j=0}^{2n-1}\left(-\frac{1}{\varkappa}\right)^j \left[(1-\varkappa^j)S(c_j) + (1+\varkappa^j)S(c'_j)\right] F_j,$$

in which

$$F_j = \sum_{k=0}^{n-1}\left(\frac{-i\nu}{\varkappa^j}\right)^k a_k, \qquad S(c) = \int_0^\infty \frac{e^{i\nu\xi}\, d\xi}{\xi - c}.$$

Let us add to the function $f(z)$ a new function

$$f_1(z) = b e^{-\frac{1}{4}\pi i(n-1)} \left[e^{\lambda_0 z} + i\operatorname{ctg}\alpha\, e^{\lambda_1 z} + \ldots\right.$$

$$\left.\ldots + i^{n-1}\operatorname{ctg}\alpha\,\operatorname{ctg}2\alpha \ldots \operatorname{ctg}(n-1)\alpha\, e^{\lambda_{n-1}z}\right],$$

which determines standing waves depending on $\sin\sigma t$. The elevation $\eta(x,t)$ of the surface for large x is found from the formula

$$\frac{2g}{\sigma}\eta(x,t) = [b\cos\tfrac{1}{4}\pi(n-1) + a\sin\tfrac{1}{4}\pi(n-1) - B]\cos(\nu x - \sigma t) +$$

$$+[a\cos\tfrac{1}{4}\pi(n-1) - b\sin\tfrac{1}{4}\pi(n-1) + A]\sin(\nu x - \sigma t) +$$

$$+[b\cos\tfrac{1}{4}\pi(n-1) - a\sin\tfrac{1}{4}\pi(n-1) + B]\cos(\nu x + \sigma t) -$$

$$-[a\cos\tfrac{1}{4}\pi(n-1) - b\sin\tfrac{1}{4}\pi(n-1) - A]\sin(\nu x + \sigma t).$$

Let us impose the additional condition: the waves formed by the source must go off to infinity. In view of this condition, the coefficients of the last two trigonometric functions must vanish. This requirement determines the two arbitrary constants a and b. With the constants a and b found in this way, the wave equation is written as follows:

$$\eta = -\frac{\sigma B}{g}\cos(\nu x - \sigma t), \qquad n \equiv 1 \pmod 4;$$

$$\eta = -\frac{\sigma}{g\sqrt{2}}(A+B)\cos(\nu x - \sigma t + \tfrac{1}{4}\pi), \qquad n \equiv 2 \pmod 4;$$

$$\eta = \frac{\sigma A}{g}\sin(\nu x - \sigma t), \qquad n \equiv 3 \pmod 4;$$

$$\eta = -\frac{\sigma}{g\sqrt{2}}(A-B)\sin(\nu x - \sigma t + \tfrac{1}{4}\pi), \qquad n \equiv 0 \pmod 4)$$

We indicate the values of the amplitudes of these waves for several special cases:

$$n = 1, \quad \alpha = 90^\circ:$$

$$-\frac{\sigma B}{g} = -\frac{2\sigma q}{g}e^{\nu y_0}\cos\nu x_0 \quad (x_0 \text{ and } y_0 \text{ are the coordinates of the source});$$

$$n = 2, \quad \alpha = 45^\circ:$$

$$-\frac{\sigma}{g\sqrt{2}}(A+B) =$$

$$= -\frac{2\sigma q}{g}\left[e^{-\nu\rho\sin\mu}\cos\left(\tfrac{1}{4}\pi + \nu\rho\cos\mu\right) + e^{-\nu\rho\cos\mu}\cos\left(\tfrac{1}{4}\pi + \nu\rho\sin\mu\right)\right];$$

$$n = 3, \quad \alpha = 30^\circ:$$

$$\frac{\sigma A}{g} = -\frac{2\sigma q}{g}\Big\{e^{-\nu\rho\sin\left(\frac{1}{3}\pi-\mu\right)}\sin\left[\nu\rho\cos\left(\tfrac{1}{3}\pi-\mu\right)\right] -$$

$$-\sqrt{3}\,e^{-\nu\rho\sin\left(\frac{1}{3}\pi+\mu\right)}\cos\left[\nu\rho\cos\left(\tfrac{1}{3}\pi+\mu\right)\right] + e^{-\nu\rho\sin\mu}\sin\left[\nu\rho\cos\mu\right]\Big\};$$

$$n = 4, \quad \alpha = 22^\circ 30' :$$

$$-\frac{\sigma}{g\sqrt{2}}(A-B) =$$

$$= \frac{8\sigma q}{g}\Big\{(1+\sqrt{2})e^{-\nu\rho\cos\mu}\sin\left(\tfrac{1}{4}\pi-\nu\rho\cos\mu\right) - e^{-\nu\rho\sin\mu}\sin\left(\tfrac{1}{4}\pi+\nu\rho\cos\mu\right) +$$

$$+(1+\sqrt{2})e^{-\nu\rho\sin\left(\frac{1}{4}\pi+\mu\right)}\sin\left[\tfrac{1}{4}\pi-\nu\rho\cos\left(\tfrac{1}{4}\pi+\mu\right)\right] - e^{-\nu\rho\sin\left(\frac{1}{4}\pi-\mu\right)}\sin\left[\tfrac{1}{4}\pi+\nu\rho\cos\left(\tfrac{1}{4}\pi-\mu\right)\right]\Big\}.$$

With the aid of these formulas one can find the geometric loci of the positions of sources that do not send progressive waves to infinity.

Received
7 V 1963

REFERENCES

1. J. V. Wehausen, E. V. Laiton, *Surface Waves*, Encyclopedia of Physics, **9**, Berlin 1960.