# AERODYNAMICS

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1963

SovietRxiv

**Abstract**

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*AERODYNAMICS*

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# SIMPLEST EXACT SOLUTIONS OF THE BOLTZMANN EQUATION FOR MOTIONS OF A RAREFIED GAS

*(Presented by Academician A. A. Dorodnitsyn, 23 X 1962)*

Motions of a monatomic gas or mixtures of monatomic gases in unbounded three-dimensional space are considered, for which the distribution function of the peculiar velocities of the particles is the same at all points. Unless otherwise stated, everywhere below we shall regard time, space coordinates, and velocities as dimensionless, referring them, respectively, to the same scales for all cases: time $t_0$, length $l_0$, velocity $l_0/t_0$. We denote by $x, y, z, t$ the Cartesian coordinates and time, and by $u, v, w$ the particle velocities; by $c$ we denote the vector with components $u, v, w$. By the distribution function $f_i(t, x, y, z, u_i, v_i, w_i)$ we shall mean the number of particles of the $i$-th species at the point $x, y, z, u_i, v_i, w_i$ of the six-dimensional space, referred to the volume element $dx\,dy\,dz\,du_i\,dv_i\,dw_i$. The Boltzmann equation has the form (1)

$$\frac{\partial f_i}{\partial t} + u_i \frac{\partial f_i}{\partial x} + v_i \frac{\partial f_i}{\partial y} + w_i \frac{\partial f_i}{\partial z} = I_i(t, r, c_i), \tag{1}$$

where $I_i$ are the collision integrals:

$$I_i = \sum_j \iiint \left[ f_i(t, r, c_i') f_j(t, r, c_j') - f_i(t, r, c_i) f_j(t, r, c_j) \right] g_{ij}\, b\, db\, d\varepsilon\, dc_j. \tag{2}$$

Here $r$ is the vector with components $x, y, z$; $c_i$ is the vector with components $u_i, v_i, w_i$; $g_{ij} = |c_i - c_j|$; $b$ is the dimensionless impact distance; $\varepsilon$ is the corresponding angle, $dc_j = du_j\,dv_j\,dw_j$.

The vectors $c_i'$, $c_j'$ of the particle velocities after collision are connected with $c_i, c_j$ by a functional dependence, determined by the nature of the interaction:

$$c_i' = \varphi_{ij}(c_i, c_j, b, \varepsilon), \qquad c_j' = \psi_{ij}(c_i, c_j, b, \varepsilon). \tag{3}$$

It is obvious that when the particles are hard elastic spheres, the similarity property holds:

$$\varphi_{ij}(\lambda c_i, \lambda c_j, b, \varepsilon) = \lambda\varphi_{ij}(c_i, c_j, b, \varepsilon); \qquad \psi_{ij}(\lambda c_i, \lambda c_j, b, \varepsilon) = \lambda\psi_{ij}(c_i, c_j, b, \varepsilon), \tag{4}$$

where $\lambda$ is an arbitrary positive quantity.

Below it is understood that, with each transition to new variables in the left-hand side of equations (1), the same transition is also made in the integrals $I_i$.

**Three-dimensional expansion–compression.** Let the macroscopic velocities be distributed according to the law

$$u = \frac{x}{t}, \qquad v = \frac{y}{t}, \qquad w = \frac{z}{t}. \tag{5}$$

These motions for the case of a non-rarefied gas were obtained by L. I. Sedov ([2]). The last expressions characterize two different types of motion. Considering them for $0 < t < \infty$, we obtain expansion, and for $-\infty < t < 0$, compression. Let us require that the equalities

$$f_i(t, x, y, z, u_i, v_i, w_i) = f_i(\tau, 0, 0, 0, U_i, V_i, W_i),$$

where $\tau = t,\ U_i = u_i - x/t,\ V_i = v_i - y/t,\ W_i = w_i - z/t;\ U_i,\ V_i,\ W_i$ are the peculiar velocities of the particles. In the variables $\tau,\ U_i,\ V_i,\ W_i$, equations (1) take the form

$$\frac{\partial f_i}{\partial \tau} - \frac{1}{\tau}\left(U_i\frac{\partial f_i}{\partial U_i} + V_i\frac{\partial f_i}{\partial V_i} + W_i\frac{\partial f_i}{\partial W_i}\right) = I_i. \tag{6}$$

For $x = y = z = 0$ we have $U_i = u_i,\ V_i = v_i,\ W_i = w_i$, and thus equations (6) are equations for the distribution functions of the absolute velocities at the point $x = y = z = 0$. This is precisely how we shall understand them below.

Let us introduce further variables $T = \tau,\ \xi_i = \tau U_i,\ \eta_i = \tau V_i,\ \zeta = \tau W_i$. In these variables equations (6) take the form

$$\frac{\partial f_i}{\partial T} = I_i. \tag{7}$$

In the trivial case of the identical vanishing of the integral $I_i$, $\partial f_i/\partial T = 0$, and we have

$$f_i = F_i(\xi_i, \eta_i, \zeta_i) = F_i(tU_i, tV_i, tW_i), \tag{8}$$

where $F_i$ are arbitrary functions of their arguments.

The total number of particles of the $i$-th kind in the volume $|t^3|$

$$|t^3|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(tU_i,tV_i,tW_i)\,dU_i\,dV_i\,dW_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(a,b,c)\,da\,db\,dc. \tag{9}$$

As it should be, it is unchanged in time. The peculiar kinetic energy of the particles of the $i$-th kind in the same volume is equal to

$$|t^3|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(tU_i,tV_i,tW_i)\frac{U_i^2+V_i^2+W_i^2}{2}\,dU_i\,dV_i\,dW_i =$$

$$= \frac{1}{t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_i(a,b,c)\frac{a^2+b^2+c^2}{2}\,da\,db\,dc. \tag{10}$$

It varies inversely proportional to $t^2$. The integrals $I_i$ are identically equal to zero in the trivial case of absence of interaction and in the case when at each instant of time the distribution of particle velocities is Maxwellian. With the latter case in mind, we shall seek a solution of the system of equations (7) in the form

$$f_i = \alpha_i \exp[k^2 m_i(\xi_i^2+\eta_i^2+\zeta_i^2)] = \alpha_i \exp[-k^2 m_{it}^2(U_i^2+V_i^2+W_i^2)]\,, \tag{11}$$

where $k$, $\alpha_i$ are constants, $m_i$ are the dimensionless masses of particles of the $i$-th kind. These expressions for the distribution functions satisfy conditions (8) and therefore, as before, make the left-hand side of equation (6) vanish. However, they also make the right-hand side of equation (6) vanish, since at each fixed instant of time they give Maxwell velocity distributions, for which the collision integrals for any potential law of interaction are equal to zero (1). Taking into account the uniqueness in time of the solution of the Boltzmann equations when the distribution functions are specified at some instant of time, we obtain:

**Theorem 1.** *If, in a three-dimensional expansion or contraction (expressed by equality (5)) of a mixture of ideal monatomic gases, the distribution of peculiar velocities is Maxwellian, then it will remain Maxwellian at every subsequent instant of time. If at $t = t_1$*

$$f_i = \alpha_i \exp\left[-k^2 m_i(U_i^2+V_i^2+W_i^2)t_i^2\right],$$

*then for $t > t_1$ equalities (11) will hold. In this case, in accordance with equalities (9), (10), the number of particles of each kind in the volume $|t|^3$ is constant,* *and the total kinetic energy of all particles in this volume, proportional to the temperature, varies inversely proportional to $t^2$.*

Let us now consider the model in which the particles are hard elastic spheres. First let us consider not a scattering-convergence motion, but an arbitrary homogeneous state of the gas, when the distribution functions $f_i = f_i^0$ in equations (1) do not depend on $r$, but depend on $t$:

$$\frac{\partial f_i^0(t, c_i)}{dt} = \sum_j \iiint \left[f_i^0(t, c_i')f_j^0(t, c_j') - f_i^0(t, c_i)f_j(t, c_j)\right] g_{ij} b\, db\, d\varepsilon\, dc_j = I_i^0(t, c_i). \tag{12}$$

Let us try to associate with each such state a certain scattering-convergence motion. We shall seek a solution of equations (7) in the form:

$$f_i = f_i^0[\chi(T), A_i], \qquad A_i = |T|C_i.$$

Here $C_i$ is the vector with components $U_i, V_i, W_i$, and the function $\chi(T)$ is to be determined. Using the similarity properties (3), (4) in transforming the right-hand side of (7), we obtain, instead of equations (7), the equations

$$\frac{d\chi(T)}{dT}\,\frac{\partial f_i^0(\chi, A_i)}{\partial\chi} = \frac{1}{T^4}\, I_i^0(\chi, A_i).$$

The equalities (12) show that, for the solution of the problem posed, one should put

$$\frac{d\chi(T)}{dT} = \frac{1}{T^4}, \qquad \chi(T) = \beta - \frac{1}{3}\frac{1}{T^3}, \qquad \beta = \text{const}.$$

Thus, to each solution $f_i = f_i^0(t, c_i)$ of the Boltzmann equations for a homogeneous state of the model of elastic spheres there corresponds the solution

$$f_i = f_i^0\left[\left(\beta - \frac{1}{3}\frac{1}{t^3}\right), |t|C_i\right]$$

of the Boltzmann equations for scattering-convergence motions. The solution in the scattering-convergence case for which, at $|t| = 1$ (this is the general case, owing to the arbitrariness of the time scale $t_0$), $f_i = \Phi_i(C_i)$, is obtained in the form

$$f_i = f_i(t, C_i) = f_i^0\left[\left(\frac{1}{3}\frac{t}{|t|} - \frac{1}{3}\frac{1}{t^3}\right), |t|C_i\right],$$

where $f_i^0(t, c_i)$ is the solution for the homogeneous state with initial data $f_i^0(0, c_i) = \Phi_i(c_i)$. In the case of scattering, $t > 0$, and as $t \to \infty$ we have

$$f_i(t, C_i) = f_i^0(1/3, tC_i),$$

i.e. in infinite time a distribution is attained that corresponds to the distribution attained for the homogeneous state already at $t = 1/3$.

For convergence motions, $t < 0$, as $t \to -0$,

$$f_i(t, C_i) = f_i^0(\infty, |t|C_i).$$

Thus, bearing in mind Theorem 1, valid for the model of hard elastic spheres, and the known property that, as $t \to \infty$, the velocity distribution in the homogeneous state tends to the Maxwell distribution (1), we obtain.

**Theorem 2.** *For the model of elastic hard spheres, in the case of scattering motions the distribution of peculiar velocities either will be the Maxwell distribution at all times, or, generally speaking, will not pass into the Maxwell distribution even as $t \to \infty$; in the case of convergence motions the distribution tends to the Maxwell distribution as $t \to -0$.*

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Received
15 X 1962

## CITED LITERATURE

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2. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Moscow, 1954.

*Note: Figure translations are in progress. See original paper for figures.*