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1963

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Abstract

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ON THE STABILITY OF CERTAIN COMPUTATIONAL ALGORITHMS INVERTING REGULAR DIFFERENCE EQUATIONS

(Presented by Academician V. I. Smirnov, 27 VI 1962)

Let the norm of a vector y^n in an n-dimensional vector space be taken to be the norm

$$\|y^n\| = \max_{i=1,\dots,n} |y_i^n|, \tag{1}$$

and let $\|l\|$ denote the norm of the matrix l, acting as a linear operator in the n-dimensional space with norm (1).

Suppose a linear algebraic system is given,

$$l^n y^n = f^n, \tag{2}$$

depending on its order n as a parameter ($n \geq N,\ N$ a fixed natural number), and suppose the matrices l^n are tridiagonal:

$$l^n = \begin{pmatrix} -b_1^n & c_1^n & & & & & \\ a_2^n & -b_2^n & c_2^n & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & a_i^n & -b_i^n & c_i^n & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & a_{n-1}^n & -b_{n-1}^n & c_{n-1}^n \\ & & & & & a_n^n & -b_n^n \end{pmatrix}.$$

The matrix l^n, depending on the parameter n, will be called a **regular difference operator** if there exist positive numbers ν and μ ($\nu < \mu$), independent of n, such that for all $n \geq N$

$$a_i^n,\ c_i^n \geq \nu, \qquad a_i^n + c_i^n \leq b_i^n, \qquad b_i^n \leq \mu, \qquad i = 1, 2, \dots, n. \tag{3}$$

In the inequalities (3) and below, by a_1^n and c_n^n we mean the expressions $b_1^n - c_1^n$ and $b_n^n - a_n^n$. Suppose the matrix of system (2) is a regular difference operator. In that case we shall call system (2) a regular difference equation (r. d. e.).

Theorem 1. For every $n \geq N$, the r. d. e. (2) is uniquely solvable.

Let ω_1 and ω_2 be arbitrary nonnegative real numbers. Denote by $\mathfrak{M}(\omega_1, \omega_2)$ the class of all regular difference operators for which the equalities

$$\min\left\{ \sup_{\substack{n\geq N\\ 1\leq i\leq j\leq n}} \prod_{k=i}^{j} \frac{a_k^n}{c_k^n}, \quad \sup_{\substack{n\geq N\\ 1\leq i\leq j\leq n}} \prod_{k=i}^{j} \frac{c_k^n}{a_k^n} \right\} = \frac{1}{\omega_1}, \qquad \inf_{\substack{n\geq N\\ 1<i<n}} (b_i^n - a_i^n - c_i^n) = \omega_2.$$

Let ω_3 be an arbitrary real number. We shall say that the R.D.E. (2) belongs to the class $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{M}(\omega_3)$, if the matrix of this system belongs to the class $\mathfrak{M}(\omega_1, \omega_2)$ and there exists a positive number K (independent of n) such that for all $n \geqslant N$, for the right-hand side of equation (2), the inequality $\|\mathbf{f}^n\| \leqslant K n^{\omega_3}$ holds.

Let $(l^n)^{-1}$ be the matrix inverse to l^n. We shall say that the order of conditioning of the class $\mathfrak{M}(\omega_1, \omega_2)$ is equal to $-\lambda$, if λ is the smallest of all numbers p such that, for each of the regular difference operators belonging to $\mathfrak{M}(\omega_1, \omega_2)$, there is a positive number C (independent of n) such that for all $n \geqslant N$ the inequality

$$\|(l^n)^{-1}\| \leqslant C n^p \tag{4}$$

is satisfied.

If inequality (4) cannot be established for any p for all operators of the class $\mathfrak{M}(\omega_1, \omega_2)$, then we shall say that the order of conditioning of the class $\mathfrak{M}(\omega_1, \omega_2)$ is equal to $-\infty$.

Theorem 2. *The order of conditioning of the class $\mathfrak{M}(\omega_1, \omega_2)$ is expressed by the value of the function $\lambda(\omega_1, \omega_2)$, which has the form*

$$\lambda(\omega_1, \omega_2) = \begin{cases} -\infty, & \omega_1 = 0, \quad \omega_2 = 0, \\ -2, & 0 < \omega_1 \leqslant 1, \quad \omega_2 = 0, \\ -1, & \omega_1 > 1, \quad \omega_2 = 0, \\ p(\omega_1, \omega_2), & \omega_2 > 0, \end{cases}$$

where $p(\omega_1, \omega_2) \geqslant 0$ for $\omega_1 > 0$.

Suppose that, in order to invert the R.D.E. (2), some computational algorithm (c.a.) Q is applied, consisting in specifying, for all $n \geqslant N$, the system of equations

$$x_k = Q_k^n(\mathbf{x}_{k-1}, \mathbf{F}), \qquad k = 1, 2, \dots, m_n, \tag{5}$$

where $\mathbf{F} = (F_1, F_2, \dots, F_{r_n})$ is a vector composed of the elements of the matrix l^n and the components of the right-hand side $\mathbf{f}^n$ of the R.D.E. (2); $\mathbf{x}_{k-1} = (x_1, x_2, \dots, x_{k-1})$; Q_k^n is some arithmetic or logical operation on certain components of the vectors $\mathbf{x}_{k-1}$ and $\mathbf{F}$. Let the values x_{k_i} $(i = 1, 2, \dots, n)$ represent the values y_i^n of the solution $\mathbf{y}^n$ of the R.D.E. (2). This form of writing a computational algorithm is used in work ([1]).

We shall say that the c.a. Q is weakly stable on the R.D.E. (2), if the corresponding system of equations (5) satisfies the following conditions:

1) there exists a positive number C_1, independent of n, such that for all $n \geqslant N$
$$|x_k| \leqslant C_1, \qquad k = 1, 2, \dots, m_n; \tag{6}$$

2) the system of equations (5) can be divided in such a way into l links
$$x_{jk} = Q^n_{jk}(\mathbf{x}_{jk-1}, \mathbf{F}_j), \qquad k = 1, 2, \dots, m_j; \quad j = 1, 2, \dots, l; \\ m_1 + m_2 + \dots + m_l = m_n, \tag{7}$$

   where $\mathbf{F}_j$ are vectors composed of certain components of the vector $\mathbf{F}$ and components of the vectors $\mathbf{x}_{sm_s}$ for $s < j$, such that if, alongside the systems (7), one considers the systems
$$x^*_{jk} = Q^n_{jk}(\mathbf{x}^*_{jk-1}, \mathbf{F}_j) + \delta^i_k \delta x_{ji}; \tag{7*}$$
$$\delta^i_k = 1 \text{ for } k = i; \qquad \delta^i_k = 0 \text{ for } k \neq i; \qquad i = 1, \dots, m_j,$$

   in which δx_{ji} in absolute value do not exceed some fi-

fixed positive number ε, then the differences $\delta x_{jk}(\delta x_{ji}) \equiv x^*_{jk} - x_{jk}$ for all $n \geqslant N$ satisfy the inequality

$$|\delta x_{jk}(\delta x_{ji})| \leqslant C_2 \varepsilon, \quad 1 \leqslant i \leqslant k \leqslant m_j; \quad j = 1, 2, \dots, l, \tag{8}$$

where the positive constant C_2 depends neither on n nor on ε.

We shall say that the c.a. Q is **strongly stable** on the r.d.e. (2) if it is weakly stable on this equation and, moreover, in the definition of weak stability the left-hand side of inequality (8) may be replaced by the quantity

$$\Delta_{jk} = \sum_{i=1}^{k} |\delta x_{jk}(\delta x_{ji})|.$$

With every c.a. Q inverting the r.d.e. (2), one can associate a c.a. $\overline{Q}$, consisting in the application of the c.a. Q to the r.d.e.

$$l^n z^n = g^n, \tag{$\overline{2}$}$$

which is obtained from system (2) if in it one reverses the numbering of the equations and unknowns, i.e. if one introduces new numbers j for the equations and unknowns of system (2) by the formula $j = n + 1 - i$, where i is the old number of an equation or unknown of system (2).

On one and the same r.d.e. one of the c.a. $Q, \overline{Q}$ may be unstable, but the other stable. Proceeding from this, we shall say that the c.a. Q is **weakly (strongly) stable in the class** $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{N}(\omega_3)$, if on each r.d.e. from this class the conditions of weak (strong) stability are satisfied by at least one of the c.a. $Q, \overline{Q}$. We shall say that the c.a. Q is **strongly unstable in the class** $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{N}(\omega_3)$, if for any number q there is an r.d.e. from this class for which there exist a positive number C and arbitrarily large indices n, for which both in the case of the c.a. Q and in the case of the c.a. $\overline{Q}$ the inequality

$$\max_{i,j,k} |\delta x_{jk}(\delta x_{ji})| \geqslant C n^q$$

is satisfied.

Consider the following computational algorithms. The formal application of the schemes of single division and of the compact scheme of the Gauss method (see ([2])) to the solution of system (2) will be called, respectively, the c.a. B_1 and the c.a. B_2. Such formal use of universal methods cannot, of course, take into account the specifics of the structure of the matrices l^n and is accompanied by a considerable number of "parasitic" arithmetic operations, the results of which can be found without computation. Freeing the c.a. B_1 and the c.a. B_2 from such operations leads to one and the same c.a. B_3

$$\sigma_i = -\frac{c_i^n}{b_i^n + a_i^n \sigma_{i-1}}, \quad i = 2, \dots, n-1; \quad \sigma_1 = -\frac{c_1^n}{b_1^n};$$

$$\psi_i = -\frac{f_i^n - a_i^n \psi_{i-1}}{b_i^n + a_i^n \sigma_{i-1}}, \quad i = 2, \dots, n; \quad \psi_1 = -\frac{f_1^n}{b_1^n};$$

$$y_i^n = \psi_i - \sigma_i y_{i+1}^n, \quad i = 1, \dots, n-1; \quad y_n^n = \psi_n.$$

An identical c.a. can also be obtained by starting from the method of difference factorization ([3]).

It can be shown that the exact values of the quantities σ_i in the computational algorithms B_1, B_2, B_3 are negative and do not exceed 1 in absolute value. We shall assume that the actually obtained values σ_i^* are not less than -1. This can always be achieved by introducing into the c.a. the operation of taking the maximum of the numbers -1 and σ_i. Then the following theorems are valid; in their formulations, by the c.a. B one may understand any of the c.a. B_1, B_2, B_3.

Theorem 3. Let $\omega_1 = 0,\ \omega_2 = 0$. The computational algorithm B is strongly unstable in the class $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{M}(\omega_3)$, whatever ω_3 may be.

Theorem 4. Let $0 < \omega_1 \leqslant 1,\ \omega_2 = 0$. In order that the computational algorithm B be weakly stable in the class $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{M}(\omega_3)$, it is necessary and sufficient that $\omega_3 \leqslant -2$.

Theorem 5. Let $\omega_1 > 1,\ \omega_2 = 0$. In order that the computational algorithm B be weakly stable in the class $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{M}(\omega_3)$, it is necessary and sufficient that $\omega_3 \leqslant -1$.

Theorem 6. Let $\omega_1 > 0,\ \omega_2 > 0,\ \omega_3 \leqslant 0$. The computational algorithm B is strongly stable in the class $\mathfrak{M}(\omega_1, \omega_2) \times \mathfrak{M}(\omega_3)$.

In the proof of Theorems 4, 5, and 6, Theorem 2 is used.

Consider on the interval $0 \leqslant x \leqslant 1$ the boundary-value problem

$$Mu \equiv (Au_x)_x + Bu_x + Cu = D, \qquad u(0) = u(1) = 0.$$

Let the coefficients A, B, C, D and the solution u be sufficiently smooth functions of the variable x, with $A(x) \geqslant \gamma > 0,\ C(x) \leqslant 0$. Let $\Delta x = \dfrac{1}{n+1},\ v_i \equiv v\left(\dfrac{i}{n+1}\right)$. Replace the original problem by the problem of solving the finite-difference equation

$$l_3^n y^n = f^n, \tag{9}$$

defined by the equalities

$$a_i^n = A_{i-1} + 4A_i - A_{i+1} - 2\Delta x B_i, \qquad b_i^n = 8A_i + 4(\Delta x)^2 C_i,$$

$$c_i^n = -A_{i-1} + 4A_i + A_{i+1} + 2\Delta x B_i; \tag{10}$$

$$f_i^n = 4(\Delta x)^2 D_i. \tag{11}$$

Theorem 7. The finite-difference equation (8) belongs to the class $\mathfrak{M}(\omega_1, 0) \times \mathfrak{M}(-2)$, $0 < \omega_1 \leqslant 1$.

From Theorems 7 and 4 it follows that the computational algorithms $B_1,\ B_2$, and B_3 are weakly stable on the finite-difference equation (8).

We now consider the boundary-value problem for the parabolic equation in the domain $[0, 1] \times [0, 1]$

$$u_t - (Au_x)_x - Bu_x - Cu = D, \qquad u(0, t) = u(1, t) = 0, \qquad u(x, 0) = 0.$$

We shall solve this problem by the method of grids, using the implicit scheme

$$\frac{u_i^{j+1}-u_i^j}{\Delta t}=\frac{a_{ij+1}u_{i-1}^{j+1}-b_{ij+1}u_i^{j+1}+c_{ij+1}u_{i+1}^{j+1}}{4(\Delta x)^2}+D_i^{j+1},$$

where $D_i^j=D(i\Delta x, j\Delta t)$, and the coefficients $a_{ij+1}, b_{ij+1}, c_{ij+1}$ are determined by the right-hand sides in formulas (10), which this time depend on the number of the time layer $j+1$. Let $\Delta x\equiv 1/(n+1)$, $\Delta t=4r(\Delta x)^2$, $r>0$. Considering n as variable, we obtain, for all j, the finite-difference equation

$$l_n^n y^n = f^n. \tag{12}$$

Theorem 8. For all j, the finite-difference equation (12) belongs to the class $\mathfrak{M}(\omega_1, 1)\times\mathfrak{M}(0)$, $0<\omega_1\leqslant 1$.

From Theorems 8 and 6 it follows that the computational algorithms B_1, B_2, and B_3 are strongly stable on the finite-difference equation (12).

The author expresses sincere gratitude to O. A. Ladyzhenskaya for supervision of the work.

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Received
20 VI 1962

REFERENCES

1. N. S. Bakhvalov, *Computational Methods and Programming*, 1, Moscow State University Press, 1962, p. 47.

2. D. K. Fadeev, V. N. Fadeeva, *Computational Methods of Linear Algebra*, 1960.

3. V. K. Saul' ev, *Integration of Equations of Parabolic Type by the Method of Grids*, 1960.

Note: Figure translations are in progress. See original paper for figures.