# MATHEMATICS

O. M. FOMENKO

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## Abstract

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MATHEMATICS

O. M. FOMENKO

# ESTIMATES OF THE PETERSSON INNER PRODUCT WITH AN APPLICATION TO THE THEORY OF QUATERNARY QUADRATIC FORMS

*(Presented by Academician I. M. Vinogradov on 5 IV 1963)*

Let $E(x_1, \dots, x_4)$ be a positive definite quaternary quadratic form with integral rational coefficients having greatest common divisor 1; let $F$ be the matrix of the form; $D$ the discriminant; $q$ the level of the form; $C_1, C_2, \dots$ absolute positive constants, all of them effective; $\tau = x + iy$ a complex variable, $y > 0$; $\Gamma(1) \supset \Gamma_0(q) \supset \Gamma(q)$ the well-known groups of integral unimodular matrices of the second order (1); $\sigma, \sigma_i, \sigma_j, \sigma_l^{(0)}$ matrices from $\Gamma(1)$; $t = \text{g.c.d.}\left(q, \frac{\bar{L}FL}{2q}\right)$, where $\bar{L}$, as well as $\bar{L}^{(1)}, \bar{L}^{(2)}, \bar{N}, \bar{G}, \bar{X}$, denotes four-dimensional integral vectors;

$$\vartheta_F(\tau/L) = \sum_N \exp\left(\pi i t \left(\bar{N} + \frac{\bar{L}}{q}\right) F \left(N + \frac{L}{q}\right)\right) = \sum_{n=0}^{\infty} \alpha_F(n, L) \exp\left(\frac{2\pi i t t n}{q}\right)$$

is a theta-series; $\vartheta_F(\tau/L) = E_F(\tau/L) + S_F(\tau/L)$ (2), where $E_F(\tau/L)$ is an Eisenstein series; $S_F(\tau/L)$ is an integral parabolic form; $\varepsilon_F(n, L)$, $\omega_F(n, L)$ are the coefficients of $\exp(2\pi i t t n/q)$ in the expansions of these forms. Obviously, $\alpha_F(n, L)$ is equal to the number of integral solutions of the equation

$$2qtn = (q\bar{X} + \bar{L})\, F\, (qX + L).$$

Recall the general fact (3) that the space of integral parabolic forms of dimension $-2k$, belonging to a subgroup $G$ of the group $\Gamma$ in (1) of finite index, is a Hilbert space of finite dimension with the Petersson inner product

$$(f, g) = \iint_{D_G} f(\tau)\overline{g(\tau)}\, y^{2(k-1)}\, dx\, dy,$$

where $D_G$ is a fundamental domain for the group $G$. The aim of the present note is to estimate certain inner products of integral parabolic forms connected with representation of a number by the form $F$, and to obtain from this some arithmetic consequences. Eichler $^{(4)}$ gave, for the remainder term $\omega_F(n, 0)$, an estimate unimprovable in the sense of $n$, showing that

$$|\omega_F(n,0)| < c_{F,\varepsilon}\, n^{1/2+\varepsilon}, \qquad (n,Q)=1.$$

This result was generalized by Shimura $^{(5)}$ and A. N. Andrianov $^{(6)}$, who showed that

$$|\omega_F(n,L)| < c_{F,L}\, \tau(n)\sqrt{n}, \qquad (n,Q)=1,$$

where $\tau(n)$ is the number of divisors of $n$; $Q$ is a certain integer. However, the dependence of the constant $c_{F,\varepsilon}$ (and $c_{F,L}$) on the form was not clarified, which in particular did not make it possible to judge starting from what point

the number $n$ is representable by the form $F$. On the basis of estimates of the Petersson inner product of integral parabolic forms we determine this dependence for the case $L = 0$. The general case is investigated analogously.

From Petersson' s results $(^7)$ it follows that the space $\mathfrak{S}\left(q, \left(\frac{D}{a}\right), q\right)$ of integral parabolic forms of type $\{\Gamma(q), -2\}$ has a basis

$$f_i(\tau) = \sum_{n=1}^{\infty} \tau_i(n) e^{2\pi i n\tau} \quad (i = 1, 2, \dots, g),$$

consisting of eigenfunctions of all Hecke operators $T_n$ $(^8)$, where $(n, q) = 1$, acting on this space, which is orthogonal, and moreover $\tau_i(1) = 1$ $(i = 1, 2, \dots, g)$. It is known $(^3)$ that $g < C_1 q^3$.

**Lemma 1.** Let $f(\tau), \varphi(\tau)$ be modular forms of dimension $-2k$; let $\Phi$ be some region in the upper half-plane of the $\tau$-plane. Then

$$\iint_{\sigma\Phi} f(\tau)\overline{\varphi(\tau)}\, y^{2(k-1)}\, dx\, dy = \iint_{\Phi} f(\tau)/\sigma \cdot \overline{\varphi(\tau)}/\sigma\, y^{2(k-1)}\, dx\, dy$$

(it is assumed that at least one of the integrals converges).

**Lemma 2.**

$$(f_i(\tau), f_i(\tau)) > \frac{1}{4\pi e^{4\pi}} q\varphi(q) \quad (i = 1, 2, \dots, g). \tag{1}$$

**Proof.** Let

$$\Gamma_0(q) = \Gamma(q)\sigma_1' \cup \Gamma(q)\sigma_2' \cup \dots \cup \Gamma(q)\sigma_{g_1}',$$

where $g_1 = q\varphi(q)$. Then (${}^3$)

$$D = \sigma {a'}_1 D^{(0)} \cup \sigma {a'}_2 D^{(0)} \cup \cdots \cup \sigma'_{g_1} D^{(0)},$$

where $D^{(0)}, D$ are fundamental regions respectively for the groups $\Gamma_0(q), \Gamma(q)$. Applying Lemma 1, we have

$$(f_i(\tau), f_i(\tau)) = \iint_D f_i(\tau)\overline{f_i(\tau)}\, dx\, dy = \sum_{\sigma {a'}_{jD}^{(0)}} \iint_{D^{(0)}} f_i(\tau)/\sigma'_j \cdot \overline{f_i(\tau)}/\sigma'_j \, dx\, dy.$$

Since $f_i(\tau)$ are forms of character $D/a$, the last sum is equal to

$$q\varphi(q) \iint_{D^{(0)}} f_i(\tau)\overline{f_i(\tau)}\, dx\, dy.$$

Further,

$$\iint_{D^{(0)}} f_i(\tau)\overline{f_i(\tau)}\, dx\, dy > \iint_{D_0} f_i(\tau)\overline{f_i(\tau)}\, dx\, dy > \frac{1}{4\pi e^{4\pi}},$$

where $D_0$ is the fundamental region for the group $\Gamma(1)$, and one may assume (${}^3$) that it consists of the points $\tau$ satisfying $|\tau| > 1,\ x < 1/2$; $D^{(0)}$ is chosen so that $D_0 \subset D^{(0)}$.

**Remark.** Apparently the stronger inequality

$$(f_i(\tau), f_i(\tau)) > C_2 q^3$$

holds, but at present we cannot obtain this result.

**Lemma 3.**

$$(S_F(\tau), S_F(\tau)) < C_3 q^7. \tag{2}$$

**Proof.** Let

$$g^2 = q^3 \prod_{p/q} \left(1 - \frac{1}{p^2}\right).$$

It is known (${}^3$) that $\Gamma(q)$ is a normal divisor of the group $\Gamma(1)$ of index $g_2$. Let

$$\Gamma(1) = \Gamma(q)\sigma_1 \cup \Gamma(q)\sigma_2 \cup \dots \cup \Gamma(q)\sigma_{g_2}, \qquad \sigma_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

$$(i = 1, 2, \dots, g_2).$$

One may assume that $1 \leqslant \gamma_i \leqslant q$. Accordingly the domain $D$ is also uniquely determined, since

$$D = \sigma_1 D_0 \cup \sigma_2 D_0 \cup \dots \cup \sigma_{g_2} D_0.$$

From the system of representatives $\sigma$ $(i = 1, 2, \dots, g_2)$ choose a complete subsystem of elements $\sigma_i^{(0)}$ that are not equivalent with respect to the equivalence relation $\sigma_{i_1}^{(0)} \sim \sigma_{i_2}^{(0)}$, which is equivalent to the equality

$$\sigma_{i_1}^{(0)} = \sigma_{i_2}^{(0)} p_k, \qquad \text{where } p_k \in \Gamma(q) \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad (k = 0, 1, \dots, q-1).$$

The number of elements in such a subsystem is

$$g_3 = q^2 \prod_{p/q} \left(1 - \frac{1}{p^2}\right).$$

Shifting $D_0$ to the right by $1, 2, \dots, q$ and uniting the resulting domains, we obtain a new domain of width $q$, which we denote by $D_1$. Let $P$ be the part of the upper half-plane of the $\tau$-plane bounded by the lines

$$x = -\frac{1}{2}, \qquad x = \frac{2q-1}{2}, \qquad y = \frac{\sqrt{3}}{2}.$$

Obviously, $D_1 \subset P$. Let

$$\sigma_i^{(0)} = \begin{pmatrix} \alpha_i^{(0)} & \beta_i^{(0)} \\ \gamma_i^{(0)} & \delta_i^{(0)} \end{pmatrix} \qquad (i = 1, 2, \dots, g_3).$$

It is easy to see that

$$(S_F(\tau), S_F(\tau)) = \sum_{i=1}^{g_3} \iint_{D_1} S_F(\tau)/\sigma_i^{(0)} \cdot \overline{S_F(\tau)/\sigma_i^{(0)}}\, dx\, dy.$$

For the form $S_F(\tau)$ the transformation formula is valid (for the corresponding formula for $\vartheta$-series, see (9)):

$$S_F(\tau)/\sigma_i^{(0)} =$$

$$= \frac{1}{(-1)(\gamma_i^{(0)})^2\sqrt{D}} \sum_{\substack{L \bmod q \\ FL \equiv 0 \ (\bmod q)}} \exp\left(-\pi i \beta_i^{(0)} \frac{\delta_i^{(0)} \overline{L} F L}{q^2}\right) \varphi(\delta_i^{(0)} L, 0)\, S(\tau/L),$$

where

$$\varphi(\delta_i^{(0)}L, 0) = \sum_{\substack{G \bmod (\gamma_i^{(0)} q) \\ G \equiv \delta_i^{(0)} L \pmod q}} \exp\left(\pi i \alpha_i^{(0)} \frac{\overline{G} F G}{\gamma_i^{(0)} q^2}\right).$$

Obviously,

$$|\varphi(\delta_i^{(0)}L, 0)| \leqslant (\gamma_i^{(0)})^4.$$

Now it is easy to obtain the inequality

$$(S_F(\tau), S_F(\tau)) \leqslant$$

$$\leqslant \sum_{\substack{\delta_i^{(0)} \\ i=1,2,\ldots,g_3}} \frac{(\gamma_i^{(0)})^4}{D} \sum_{\substack{L^{(1)} \bmod q \\ FL^{(1)} \equiv 0 \ (\bmod q)}} \sum_{\substack{L^{(2)} \bmod q \\ FL^{(2)} \equiv 0 \ (\bmod q)}} \left| \iint_P S(\tau/L^{(1)}) \overline{S(\tau/L^{(2)})}\, dx\, dy \right|.$$

It remains to estimate

$$\iint_P S(\tau/L^{(1)}) \overline{S(\tau/L^{(2)})}\, dx\, dy.$$

We have

$$S(\tau/L) = \sum_{n=1}^{\infty} \omega_F(n, L) \exp\left(\frac{2\pi i n \tau}{q}\right).$$

But

$$\omega_F(n, L) = \alpha_F(n, L) - \varepsilon_F(n, L).$$

Proceeding from geometric considerations, one easily obtains the estimate

$$\alpha_F(n, L) < C_4 \frac{n^2 t^2}{q^2 \sqrt{D}}.$$

Using the basis of the space of Eisenstein series of dimension $-2$, which Hecke obtained explicitly (2), it is easy to show that

$$\varepsilon_F(n, L) < C_5 \frac{ntq}{\sqrt{D}},$$

whence, for ⋯

For $n > C_6 q^4$ we obtain

$$|\omega_F(n, L)| < C_7 \frac{n^2 t^2}{q^2 \sqrt{D}}.$$

With the aid of this estimate we prove that

$$\left| \iint_{\dot{P}} S(\tau/L^{(1)}) \overline{S(\tau/L^{(2)})}\, dx\, dy \right| < C_8 \frac{q}{D}. \tag{3}$$

Using the lemma on the number of solutions of the congruence $FL \equiv 0 \pmod{q}$ ($^{10}$) and (3), we easily obtain (2).

**Theorem 1.**

$$|\omega_F(n,0)| < C_9 q^4 \ln\ln q \sqrt{n}\, \tau(n). \tag{4}$$

**Proof.** Let

$$S_F(\tau) = \sum_{i=1}^{g} \alpha_i f_i(\tau).$$

From the results of papers ($^{4-6,11}$) it follows that

$$|\tau_i(n)| \ll \tau(n)\sqrt{n}, \qquad (n,q) = 1.$$

Since

$$(S_F(\tau), S_F(\tau)) = \sum_{i=1}^{g} |\alpha_i|^2 (f_i(\tau), f_i(\tau)),$$

the assertion of the theorem follows from Lemmas 2 and 3.

**Theorem 2.** *Let $n > C_{10} D^{14.01}$ be an odd number and let certain natural congruence conditions be satisfied (see ($^{12}$)). Then $n$ is representable by the form $F$, and for the number of representations an asymptotic formula holds.*

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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*Note: Figure translations are in progress. See original paper for figures.*

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