THE FIELD EMBEDDING PROBLEM

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Abstract

Full Text

MATHEMATICS

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THE FIELD EMBEDDING PROBLEM

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Let

$$1 \to \mathfrak{A} \to \mathfrak{G} \xrightarrow{\pi} \mathfrak{F} \to 1$$

be an exact sequence of groups, where $\mathfrak{A}$ is an abelian group of period n, and k is a finite normal extension of the field k_0 with group $\mathfrak{F}$, containing a primitive n-th root of unity. It is required to construct a field (or Galois algebra) K, containing k and normal over k_0 with group $\mathfrak{G}$, such that for $g \in \mathfrak{G}$ we have $g|_k = \pi(g)$. Arguing analogously to ([2]), one can reduce this problem to the following one.

Let C be the multiplicative group of the field k (which we shall write additively), $\overline{\mathfrak{A}} = \mathrm{Hom}_Z(\mathfrak{A}, C)$, and suppose that an operator f from $\mathfrak{F}$ acts on an element $\overline{a} \in \overline{\mathfrak{A}}$ by the formula

$$(\overline{a}f)(a) = [\overline{a}(faf^{-1})]f$$

for every $a \in \mathfrak{A}$. The groups C and $\overline{\mathfrak{A}}$, by means of π, are $\mathfrak{G}$-modules. It is required to construct a right $\mathfrak{G}$-module X such that: 1) the sequence of $\mathfrak{G}$-modules

$$0 \to C \xrightarrow{\psi} X \xrightarrow{\varphi} \overline{\mathfrak{A}} \to 0$$

is exact; 2) for any $a \in \mathfrak{A}$, $x \in X$, we have the equality

$$xa = x + \psi[\varphi(x)(a)].$$

Condition 1) means that X is an extension of $\overline{\mathfrak{A}}$ by means of C and may be specified as follows. Let $\beta : P \to \overline{\mathfrak{A}}$ be an epimorphism, where P is a $\mathfrak{G}$-projective module. Denote by $Z(\overline{\mathfrak{A}})$ the integral group module for $\overline{\mathfrak{A}}$, with the operators from $\mathfrak{F}$ permuting the generators of this module in the same way as the corresponding elements of $\overline{\mathfrak{A}}$. For the sequel, we choose P so that β is the composition of epimorphisms

$$\beta_1 : P \to Z(\overline{\mathfrak{A}}), \qquad \beta_2 : Z(\overline{\mathfrak{A}}) \to \overline{\mathfrak{A}},$$

where β_2 assigns to the generators of the module $Z(\overline{\mathfrak{A}})$ the corresponding elements of $\overline{\mathfrak{A}}$. Denote $\operatorname{Ker}\beta$ by M. X is completely determined by an element

$$\gamma \in \operatorname{Hom}_{\mathfrak{G}}(M, C).$$

Condition 2) means that

$$\gamma[p(a-1)] = (\beta p)(a) \qquad (p \in P,\ a \in \mathfrak{A}).$$

Let M_0 be the $\mathfrak{G}$-submodule of M generated by all elements $p(a-1)$. Thus, the restriction of γ to M_0 must coincide with the homomorphism known to us, which we denote by γ_1.

There is a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
M_0 & & = M_0 \\
\downarrow & & \downarrow \\
0 \to M & & \to P \xrightarrow{\beta} \overline{\mathfrak{A}} \to 0 \\
\downarrow & & \downarrow \\
0 \to M/M_0 & & \to P/M_0 \xrightarrow{\overline{\beta}} \overline{\mathfrak{A}} \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

with naturally defined homomorphisms. It induces the following diagram with exact rows and a commutative triangle:

$$\begin{array}{c}
\operatorname{Hom}_{\mathfrak{G}}(M, C) \\
\downarrow \alpha_1 \\
\operatorname{Hom}_{\mathfrak{G}}(M_0, C) \\
\swarrow \alpha_3 \qquad \downarrow \alpha_2 \\
\operatorname{Ext}^1_{\mathfrak{G}}(\overline{\mathfrak{A}}, C) \xrightarrow{\alpha_4} \operatorname{Ext}^1_{\mathfrak{G}}(P/M_0, C) \xrightarrow{\alpha_5} \operatorname{Ext}^1_{\mathfrak{G}}(M/M_0, C)
\end{array}$$

From the preceding it follows that, for solvability of the embedding problem, it is necessary and sufficient that there exist a

$$\gamma \in \operatorname{Hom}_{\mathfrak{G}}(M, C)$$

such that

$$\alpha_1\gamma = \gamma_1.$$

The last diagram makes it possible to give two more forms of this condition: 1) $\alpha_2\gamma_1 = 0$, or, what is the same, $\alpha_5\gamma_2 = 0$, where $\gamma_2 = \alpha_3\gamma_1 \in \mathrm{Ext}^1_{\mathfrak{G}}(P/M_0, C)$; 2) there exists $\gamma_3 \in \mathrm{Ext}^1_{\mathfrak{G}}(\overline{\mathfrak{A}}, C)$ for which $\alpha_4\gamma_3 = \gamma_2$.

The homomorphism $\overline{\beta} : P/M_0 \to \overline{\mathfrak{A}}$ can be represented as a composite homomorphism

$$P/M_0 \to Z(\overline{\mathfrak{A}}) \to \overline{\mathfrak{A}},$$

since $M_0 \subset \mathrm{Ker}\,\beta$. Therefore α_4 is represented as a composite homomorphism

$$\mathrm{Ext}^1_{\mathfrak{G}}(\overline{\mathfrak{A}}, C) \xrightarrow{\alpha_6} \mathrm{Ext}^1_{\mathfrak{G}}(Z(\overline{\mathfrak{A}}), C) \xrightarrow{\alpha_7} \mathrm{Ext}^1_{\mathfrak{G}}(P/M_0, C).$$

The existence of $\gamma_4 \in \mathrm{Ext}^1_{\mathfrak{G}}(Z(\overline{\mathfrak{A}}), C)$ such that $\alpha_7\gamma_4 = \gamma_2$ (one can see that γ_4 is then determined uniquely) is equivalent to the compatibility condition (1^2).

The additional condition for embedding is that there exists

$$\gamma_3 \in \mathrm{Ext}^1_{\mathfrak{G}}(\overline{\mathfrak{A}}, C)$$

for which $\alpha_6\gamma_3 = \gamma_4$. Denote $\mathrm{Ker}\,\beta_2$ by D. We have the exact sequence

$$\mathrm{Ext}^1_{\mathfrak{G}}(\overline{\mathfrak{A}}, C) \xrightarrow{\alpha_6} \mathrm{Ext}^1_{\mathfrak{G}}(Z(\overline{\mathfrak{A}}), C) \xrightarrow{\alpha_8} \mathrm{Ext}^1_{\mathfrak{G}}(D, C).$$

Therefore the additional condition takes the form

$$\gamma_5 = \alpha_8\gamma_4 = 0.$$

But

$$\mathrm{Ext}^1_{\mathfrak{G}}(D, C) = H^1(\mathfrak{G}, \mathrm{Hom}_Z(D, C))$$

(see (4), Theorem XI.9.2). Let $\mathfrak{G}_0 \supset \mathfrak{A}$ be a normal subgroup of $\mathfrak{G}$ consisting of all elements acting trivially on $\overline{\mathfrak{A}}$; $\overline{\mathfrak{F}} = \mathfrak{G}/\mathfrak{G}_0$; C_0 the subfield of C belonging to $\pi(\mathfrak{G}_0)$. The exact sequence

$$0 \to H^1(\overline{\mathfrak{F}}, \mathrm{Hom}_Z(D, C_0)) \xrightarrow{\lambda} H^1(\mathfrak{G}, \mathrm{Hom}_Z(D, C)) \xrightarrow{i} H^1(\mathfrak{G}_0, \mathrm{Hom}_Z(D, C)),$$

where λ is the lift homomorphism and i the restriction homomorphism. $i\gamma_5 = 0$, since this equality is equivalent to the solvability of the embedding problem corresponding to the sequence

$$1 \to \mathfrak{A} \to \mathfrak{G}_0 \to \pi(\mathfrak{G}_0) \to 1,$$

and this problem is Brauerian. Therefore

$$\gamma_5 = \lambda\delta, \qquad \delta \in H^1(\overline{\mathfrak{F}}, \mathrm{Hom}_Z(D, C)) = \mathrm{Ext}^1_{\overline{\mathfrak{F}}}(D, C_0).$$

Thus we have proved

Theorem 1. If the compatibility condition is fulfilled, then for embeddability it is necessary and sufficient in addition that the element δ of $\mathrm{Ext}^1_{\overline{\mathfrak{F}}}(D, C_0)$ be equal to 0.

Theorem 2. If $\overline{\mathfrak{F}} = \langle f \rangle$ is a cyclic group, then compatibility is sufficient for embeddability.

The proof is based on a chain of lemmas, most of which are only formulated here. By the theorems of Kochendörffer ([3]) it is enough to consider $\overline{\mathfrak{F}}$ and $\overline{\mathfrak{A}}$ as p-groups. The group $H^0(\overline{\mathfrak{F}}, C_0)$ is Abelian and $(\overline{\mathfrak{F}} : 1)$ is its period. By Prüfer' s theorem (see, for example, ([6]), p. 156), it is a direct sum of cyclic groups $\{a_i\}$, $i \in I$. By a_i denote the invariant element of C_0 corresponding to $\overline{a}_i$. Let n_i be the order of $\overline{a}_i$, $n_i m_i = (\overline{\mathfrak{F}} : 1)$.

Lemma 1. For every $i \in I$ there exists $c_i \in C_0$, $c_i(f^{m_i} - 1) = 0$,

$$a_i = c_i(1 + f + \cdots + f^{m_i - 1}).$$

Next, consider the $\overline{\mathfrak{F}}$-module

$$B = C \oplus \sum \oplus Z(\overline{\mathfrak{F}}) e_i,$$

where $Z(\overline{\mathfrak{F}})$ is the integral group module of the group $\overline{\mathfrak{F}}$, and its $\overline{\mathfrak{F}}$-submodule B_0, generated by the elements

$$c_i = e_i(1 + f^{m_i} + \cdots + f^{m_i(n_i - 1)}).$$

Obviously,

$$B_0 \simeq \sum \oplus Z(\overline{\mathfrak{F}} / \langle f^{m_i} \rangle).$$

Lemma 2. $H^1(\overline{\mathfrak{F}}_1, B/B_0) = 0$ for every subgroup $\overline{\mathfrak{F}}_1$ of the group $\overline{\mathfrak{F}}$;

$$H^2(\overline{\mathfrak{F}}, B/B_0) = 0.$$

The proof of this lemma is quite cumbersome.

Lemma 3. Let Z_p be a cyclic group of order p, on which the operators from $\overline{\mathfrak{F}}$ act trivially. Then

$$\mathrm{Ext}^2_{\overline{\mathfrak{F}}}(Z_p, B/B_0) = 0.$$

Proof. The exact sequence

$$\mathrm{Ext}^1_{\overline{\mathfrak{F}}}(pZ, B/B_0) \to \mathrm{Ext}^2_{\overline{\mathfrak{F}}}(\mathfrak{A}, B/B_0) \to \mathrm{Ext}^2_{\overline{\mathfrak{F}}}(Z, B/B_0).$$

By Lemma 2 the end terms are equal to 0.

Lemma 4.

$$\mathrm{Ext}^2_{\overline{\mathfrak{F}}}(\overline{\mathfrak{A}}, B/B_0) = 0.$$

It is proved by induction on $(\overline{\mathfrak{A}} : 1)$ with the aid of Lemma 3.

Lemma 5.
$$\mathrm{Ext}^1_{\mathfrak{F}}(D, B/B_0) = 0.$$

Proof. The sequence

$$\mathrm{Ext}^1_{\mathfrak{F}}(Z(\overline{\mathfrak{A}}), B/B_0) \longrightarrow \mathrm{Ext}^1_{\mathfrak{F}}(D, B/B_0) \longrightarrow \mathrm{Ext}^2_{\mathfrak{F}}(\overline{\mathfrak{A}}, B/B_0)$$

is exact. Its first term is equal to 0 by the first assertion of Lemma 2, and the last one by Lemma 4.

Lemma 6. $\mathrm{Ext}^1_{\mathfrak{F}}(D, B_0) = 0.$

Lemma 7. $\mathrm{Ext}^1_{\mathfrak{F}}(D, C) = \mathrm{Ext}^1_{\mathfrak{F}}(D, B) = 0.$

The assertion of Theorem 2 follows from Lemma 7.

It is proved similarly that, if k is a local field, then consistency is sufficient for embeddability (5).

The same methods make it possible to describe the set of solutions of the embedding problem:

Theorem 3. *If $\mathfrak{G}$ is a semidirect extension of $\mathfrak{F}$ by means of $\mathfrak{A}$, then there exists a one-to-one correspondence between the solutions of the embedding problem and the elements of the group* $\mathrm{Ext}^1_{\mathfrak{F}}(\overline{\mathfrak{A}}, C)$.

Indeed, let the $\mathfrak{F}$-module X be an extension of $\overline{\mathfrak{A}}$ by means of C. Put

$$xa = x + \psi[\varphi(x)(a)], \qquad x \in X, \quad a \in \mathfrak{A}.$$

Then, as is easy to see, X is a $\mathfrak{G}$-module satisfying conditions 1) and 2), which were formulated at the beginning.

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Note: Figure translations are in progress. See original paper for figures.