MATHEMATICS

O. D. TSERETELI

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

O. D. TSERETELI

FOURIER SERIES AND METRIC PROPERTIES OF FUNCTIONS*

(Presented by Academician V. I. Smirnov on 5 II 1963)

Let G be the set of all measure-preserving Lebesgue transformations of the interval $(0, 2\pi)$ onto itself that are invertible. Following V. A. Rokhlin ([1]), measurable functions $f(x)$ and $f_1(x)$, $x \in (0, 2\pi)$, will be called **metrically equivalent** on $(0, 2\pi)$ if there exists an $\omega \in G$ such that $f(x) = f_1[\omega(x)]$ almost everywhere on $(0, 2\pi)$. A property of a function $f(x)$ that is shared by all functions metrically equivalent to it will be called its **metric property**. Such properties include, for example, the property of summability of $f(x)$ on $(0, 2\pi)$.

It is of interest to ask whether various properties of summable functions, connected with their Fourier series, are metric. For example, is it a metric property of a function $f(x)$, summable on $(0, 2\pi)$, to have a Fourier series $\sigma(f)$ that diverges on a set of positive measure (even at a single point)? From Theorems 1 and 3, formulated below, it follows that these properties are not metric.

Further, is the property that the series $\sigma(f)$ converges almost everywhere a metric property? From Theorem 1 and from the fact that there exists a summable function whose Fourier series diverges almost everywhere, it follows that for the class of all summable functions the indicated property is not metric. For a narrower class of functions, for example for $L_2(0, 2\pi)$, the question remains open, and a positive solution of it, in view of Theorem 1, is equivalent to confirmation of the well-known hypothesis of N. N. Luzin that the series $\sigma(f)$ for $f \in L_2(0, 2\pi)$ converges almost everywhere.**

If one asks whether convergence everywhere, with the possible exception of some set of the first category, of the series $\sigma(f)$ is a metric property of a function $f \in L(0, 2\pi)$, then the question is settled by Theorem 3, which asserts that if $f \in L(0, 2\pi)$ is not equivalent to a constant on $(0, 2\pi)$, then there exists f_1, metrically equivalent to f, such that $\sigma(f_1)$ diverges on a set of the second category.

If the set G is endowed with a certain topology T, the question of the stability of various properties of measurable functions in general, and in particular (in the case of summable functions) those connected with their Fourier series, becomes of interest. Namely, if $f(x)$ has some property, will $f[\omega(x)]$ have the same property for all ω belonging to some neighborhood O_T of the identity transformation

$1(x) \equiv x$?

We shall consider two topologizations of the set G, due to P. R. Halmos ([3]). These are the weak and the uniform topologies, denoted by us respectively by W and U. The topology W is defined by a system of neighborhoods in the following way. Let e_i, $i = 1, 2, \dots, n$, be arbitrary measurable subsets of $(0, 2\pi)$, and let $\varepsilon > 0$.

The neighborhood $O_W(\omega_0) = O^*_W(\omega_0; e_1, \dots, e_n; \varepsilon)$ of the transformation $\omega_0 \in G$ is

---

* Theorems 1 and 2 of the present note were announced in a report on 10 VII 1961 at the IV All-Union Mathematical Congress.

** In a note by Z. Zagorski ([2]) the validity of N. P. Luzin's hypothesis is asserted; however, a proof of this assertion has not yet appeared in print.

the set of all $\omega \in G$ satisfying the inequalities

$$\mu\left[v_0(\varepsilon_i)\Delta\omega(\varepsilon_i)\right] < \varepsilon, \qquad i = 1, 2, \dots, n,$$

where μ is Lebesgue measure and Δ denotes the sign of the symmetric difference of sets. The uniform topology U is defined by means of the metric

$$\rho(\omega_1, \omega_2) = \mu\{x : \omega_1(x) \neq \omega_2(x)\}, \qquad \omega_1,\ \omega_2 \in G.$$

The topological space obtained by endowing the set G with the topology $W(U)$ will be denoted by (G, W) $((G, U))$.

Theorem 1. *Let $f(x) \in L(0, 2\pi)$. Then the set of those $\omega \in G$ for which the series $\sigma(f\omega)$ converges almost everywhere is everywhere dense in the space (G, W).*

Thus this theorem asserts that the property of a function $f \in L(0, 2\pi)$ of having a Fourier series that diverges on a set of positive measure is not stable in the sense of the topology W.

The interesting question of whether the same is true in the sense of the uniform topology remains open in the general case. We note that a positive solution of this question would mean a strengthening, in a certain sense, of the well-known theorems of D. E. Men' shov ([4,5]) on the "correction" of functions. The question can be answered positively for one special class of functions, namely for elementary summable functions; here a measurable function defined on a measurable set E is called elementary if on this set it is equivalent to a function whose set of values is at most countable.

Theorem 2. *Let $f(x)$ be an elementary summable function on $(0, 2\pi)$. Then the set of those $\omega \in G$ for which the series $\sigma(f\omega)$ converges almost everywhere is everywhere dense in the space (G, U).*

It should be noted that there exist elementary summable functions, defined on $(0, 2\pi)$, whose Fourier series diverge almost everywhere. This result can be obtained by modifying A. N. Kolmogorov' s example of a summable function with an almost everywhere divergent Fourier series.

Theorem 3. *Let $f(x) \in L(0, 2\pi)$ and let it not be equivalent to a constant on $(0, 2\pi)$. Then the set of those $\omega \in G$ for which the series $\sigma(f\omega)$ diverges unboundedly on a set everywhere dense and of type G_δ is everywhere dense in the space (G, U).*

This theorem resolves the question of stability, in the sense of the topology U, of the property of a summable function of having a Fourier series convergent everywhere, with the possible exception of some set of the first category. This property turns out to be stable only for functions equivalent on $(0, 2\pi)$ to constants.

Tbilisi Mathematical Institute
named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

Received
27 I 1963

CITED LITERATURE

1. V. A. Rokhlin, *Uspekhi Mat. Nauk*, **12**, No. 2, 167 (1957).

2. Z. Zahorski, *C. R.*, No. 16, 2366 (1961).

3. P. R. Halmos, *Lectures on Ergodic Theory*, IL, 1959.

4. D. E. Men' shov, *Mat. sb.*, **8** (50), 493 (1940).

5. D. E. Men' shov, *Tr. Mosk. matem. obshch.*, **1**, 5 (1952).

Note: Figure translations are in progress. See original paper for figures.