



Soviet-era science, translated into English

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1963

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Abstract

Full Text

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ON UNIVERSAL SEQUENCES OF FUNCTIONS

A sequence of measurable functions

$$f_m(x) \quad (m = 0, 1, 2, \dots), \quad (1)$$

defined almost everywhere on some interval $[a, b]^*$, will be called **universal** on $[a, b]$ if, for every measurable function $f(x)$ defined almost everywhere on $[a, b]$, one can choose an increasing sequence of natural numbers m_k ($k = 0, 1, 2, \dots$) such that

$$\lim_{k \rightarrow \infty} f_{m_k}(x) = f(x) \quad (2)$$

almost everywhere on $[a, b]$.

We next introduce the definition of a sequence universal with respect to two functions $F_1(x)$ and $F_2(x)$, or of a sequence of type $A(F_1, F_2)$.**

Let the functions $F_1(x)$ and $F_2(x)$ be measurable, defined almost everywhere on the interval $[a, b]$, and satisfy the inequality

$$F_1(x) \leq F_2(x) \quad (3)$$

almost everywhere on this interval. We shall say that the sequence (1) of measurable functions, defined almost everywhere on $[a, b]$, is a **sequence of type** $A(F_1, F_2)$ if, for every measurable function $f(x)$ defined almost everywhere on $[a, b]$ and satisfying the inequality

$$F_1(x) \leq f(x) \leq F_2(x) \quad (4)$$

almost everywhere on the same interval, one can determine an increasing sequence of natural numbers m_k ($k = 0, 1, 2, \dots$) such that equality (2) will hold almost everywhere on $[a, b]$ ((¹), p. 3).

Let us introduce one more definition. Let the functions $F_1(x)$ and $F_2(x)$ satisfy the same conditions as in the preceding definition. We shall say that the sequence (1) of measurable functions, defined almost everywhere on $[a, b]$, is a

sequence of type $B(F_1, F_2)$ if, for any four measurable functions $\varphi_1(x)$, $\psi_1(x)$, $\psi_2(x)$, $\varphi_2(x)$, defined almost everywhere on $[a, b]$ and satisfying the inequality

$$F_1(x) \leq \varphi_1(x) \leq \psi_1(x) \leq \psi_2(x) \leq \varphi_2(x) \leq F_2(x) \quad (5)$$

almost everywhere on the same interval, one can determine an increasing sequence of natural numbers m_k ($k = 0, 1, 2, \dots$) such that

$$\alpha^0. \quad \lim_{k \rightarrow \infty} f_{m_k}(x) = \varphi_1(x), \quad \overline{\lim}_{k \rightarrow \infty} f_{m_k}(x) = \varphi_2(x)$$

almost everywhere on $[a, b]$.

$$\beta^0. \quad \psi_1(x) \text{ and } \psi_2(x)$$

are respectively the lower and upper limits in measure on $[a, b]$ of the sequence of functions

$$f_{m_k}(x) \quad (k = 0, 1, 2, \dots). \quad (6)$$

(The definition of upper and lower limits in measure is given in ⁽²⁾, p. 4.)

* The functions of the sequence (1) may be equal to $+\infty$ or $-\infty$ on sets of positive measure.

** In what follows, when speaking of universal sequences, we shall not mention the interval on which they are universal.

An infinite series whose terms are measurable functions, finite almost everywhere on $[a, b]$, will be called **universal** (respectively, a series of type $A(F_1, F_2)$ or of type $B(F_1, F_2)$) if the sequence of its partial sums is universal (respectively, of type $A(F_1, F_2)$ or of type $B(F_1, F_2)$).

In ⁽¹⁾ the following theorem was proved.

If the functions $F_1(x)$ and $F_2(x)$ satisfy the preceding conditions, where $[a, b] = [-\pi, \pi]$, then the classes of trigonometric series of type $A(F_1, F_2)$ and of type $B(F_1, F_2)$ coincide (⁽¹⁾, Theorem 1, p. 5).

It turns out that this theorem remains valid for arbitrary sequences of measurable functions. One can even prove a more general assertion. To formulate this assertion we introduce the definition of sequences of type $C(F_1, F_2)$.

First of all we introduce the definition of a limit function of a sequence. Let the sequence (1) satisfy the previous conditions. A function $\varphi(x, E)$, defined almost everywhere on the set E , $\text{mes } E > 0$, $E \subset [a, b]$, will be called a **limit function**

of the sequence (1) if it is possible to define an increasing sequence of natural numbers n_ν ($\nu = 0, 1, 2, \dots$) such that

$$\lim_{\nu \rightarrow \infty} f_{n_\nu}(x) = \varphi(x, E)$$

almost everywhere on E .

Suppose further that $M = \{\varphi(x, E)\}$ is some set of functions, each of which is defined almost everywhere on the corresponding set E , $\text{mes } E > 0$, $E \subset [a, b]$, and take two measurable functions $F_1(x)$ and $F_2(x)$ satisfying inequality (3) almost everywhere on $[a, b]$. We shall say that the set M **has the property** $h(F_1, F_2)$ if there exists a sequence of measurable functions, defined almost everywhere on $[a, b]$, for which M is the set of all limit functions, and $F_1(x)$ and $F_2(x)$ are respectively its lower and upper limits in measure on $[a, b]$.

A **sequence of type** $C(F_1, F_2)$ will mean a sequence (1) of measurable functions, defined almost everywhere on $[a, b]$, which satisfies the following condition:

For any four measurable functions $\varphi_1(x), \psi_1(x), \psi_2(x), \varphi_2(x)$, satisfying inequalities (5) almost everywhere on $[a, b]$, and for any set $M = \{\varphi(x, E)\}$ possessing the property $h(\psi_1, \psi_2)$, it is possible to define an increasing sequence of natural numbers m_k ($k = 0, 1, 2, \dots$) such that M is the set of all limit functions of the sequence (6) and, moreover, the equalities α^0 and the condition β^0 hold.

It is clear that every sequence of type $C(F_1, F_2)$ is also a sequence of type $B(F_1, F_2)$.

Theorem 1. *The class of sequences of type $A(F_1, F_2)$ coincides with the class of sequences of type $C(F_1, F_2)$.*

Hence, as a consequence, we obtain

Theorem 2. *The class of sequences of type $A(F_1, F_2)$ coincides with the class of sequences of type $B(F_1, F_2)$.*

The last theorem is a generalization of the above-mentioned theorem on trigonometric series.

We shall now consider sequences that are universal with respect to some summability method.

Take a method T , defined by the matrix $\|a_{mk}\|$ ($m, k = 0, 1, 2, \dots$). We shall say that the sequence (1) of measurable functions, finite almost everywhere on some segment $[a, b]$, is a **universal sequence with respect to the method** T , if for every measurable function, defined almost everywhere on $[a, b]$, one can choose such an increasing sequence of natural numbers

numbers m_k ($k = 0, 1, 2, \dots$), such that the sequence (6) is summable by the method T to $f(x)$ almost everywhere on $[a, b]$.

If T is a completely regular method*, then it is obvious that any sequence satisfying the preceding conditions and universal in the ordinary sense will also be universal with respect to the method T .

It turns out that the latter assertion can be extended to methods T that transform every sequence converging to zero into a sequence converging to zero, if one additionally requires that these methods be normal, i.e., such that

$$a_{mk} = 0 \quad (k > m), \quad a_{mm} \neq 0 \quad (m = 0, 1, 2, \dots),$$

where a_{mk} are the elements of the matrix defining the method T under consideration.

Received
13 V 1963

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- ³ G. Hardy, *Divergent Series*, IL, 1951.

* The definition of a completely regular method is given in (3) (1,4, p. 24).

Note: Figure translations are in progress. See original paper for figures.

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