L. N. Prokopenko

1963

SovietRxiv

Abstract

Full Text

L. N. Prokopenko

On the Uniqueness of the Solution of the Cauchy Problem for Operator-Differential Equations

(Presented by Academician I. G. Petrovskii on 21 VIII 1962)

In the present note we consider the question of uniqueness of the solution of the Cauchy problem for the operator-differential equation in a Banach space

$$\frac{du}{dt} = Au \qquad (0 \leqslant t < T \leqslant \infty);$$

$$u\,\Big|_{t=0} = 0. \tag{1}$$

(and also for more general equations). The results set forth below are adjacent to the investigations of Yu. I. Lyubich (^{1,2}) and constitute their further development.

1°. In equation (1) the operator A is a closed linear operator in a complex Banach space B (moreover, its domain of definition D_A is not assumed, in general, to be dense in B). By a solution of the Cauchy problem (1) we shall understand a vector-function $u = u(t)$, $0 \leqslant t < T$, with values in B, which almost everywhere satisfies (1) in the weak sense (i.e. $u(t)$ is (weakly) absolutely continuous, $u'(t) = du/dt$ exists almost everywhere (in the weak sense), for almost all t one has $u(t) \in D_A$ and $u'(t) = Au(t)$, $\lim_{t\to 0} u(t) = 0$ (in the weak sense)).

Condition A. There exist: a) a ray l in the complex plane:

$$l = \{z : z = re^{i\varphi} + z_0, \quad 0 \leqslant r < \infty\},$$

forming an acute angle with the positive direction of the real axis (i.e. $|\varphi| < \pi/2$)*; b) constants $\sigma > 0$ and $C > 0$ such that for $u \in D_A$ and $z \in l$

$$\|Au - zu\| \geqslant Ce^{-\sigma \operatorname{Re} z}\|u\|. \tag{2}$$

Theorem 1. *If Condition A is fulfilled and $\sigma < T < \infty$, then the solution of the Cauchy problem (1) is identically equal to zero on $[0, T - \sigma]$.*

Let us indicate the idea of the proof. Consider the Laplace transform of $u(t)$

$$\hat{u}(z) = \int_0^T e^{-zt} u(t)\, dt$$

(the integral is understood in the sense of Bochner); $\hat{u}(z)$ is an entire vector-function of finite degree, and, moreover ($x = \operatorname{Re} z$),

$$\|\hat{u}(z)\| \leqslant \mathrm{const} \cdot \frac{1 - e^{-xT}}{x}. \tag{3}$$

Using equation (1), one can show that $\hat{u}(z) \in D_A$ for all z and

$$A\hat{u}(z) - z\hat{u}(z) = e^{-zT} u(T).$$

Hence, by Condition A, it follows that for $z \in l$

$$\|\hat{u}(z)\| \leqslant C_1 e^{\sigma x} \|A\hat{u}(z) - z\hat{u}(z)\| = C_2 e^{x(\sigma - T)}. \tag{4}$$

* Such a ray will henceforth be called positive.

Consider in the complex plane the straight line $l_1 = \{z : \operatorname{Re} z = \operatorname{Re} z_0\}$ and the angles K_1 and K_2 between l_1 and the ray l. The entire function $U(z) = e^{z(T-\sigma)}\hat{u}(z)$, by virtue of inequalities (3) and (4), is bounded on l and l_1, i.e., on the sides of the angles K_1 and K_2; on the basis of the Phragmén–Lindelöf principle it proves to be bounded also inside the angles K_1 and K_2. Thus, throughout the half-plane $\operatorname{Re} z \geqslant \operatorname{Re} z_0$ the estimate $\|\hat{u}(z)\| \leqslant \mathrm{const}\, e^{-x(T-\sigma)}$ ($x = \operatorname{Re} z$) holds. Hence we obtain that $u(t) = 0$ for $0 \leqslant t \leqslant T - \sigma$ (cf., for example, ([3]), pp. 175-176; ([4]), pp. 345-346).

Corollary 1. *If condition A is satisfied with some* $\sigma > 0$*, then there is no nonzero solution of the Cauchy problem (1) defined for all* $t > 0$ *(in other words, global uniqueness holds).*

This follows from the fact that in Theorem 1, T, and consequently also $T - \sigma$, may be arbitrary numbers.

Corollary 2. *If condition A is satisfied for arbitrary* $\sigma > 0$ *(with* C *depending on* σ*), then uniqueness holds for the Cauchy problem (1) on any interval* $[0, T]$ *(local uniqueness).*

Indeed, in this case $T - \sigma$ may be arbitrarily close to T.

Theorem 1 was essentially proved by Yu. I. Lyubich under the assumption that along the ray l there exists the resolvent $R_z = (A - zE)^{-1}$. This assumption is essential for the method of proof proposed by him. In this case inequality (2) may be written in the following equivalent form:

$$\|R_z\| \leqslant Ce^{\sigma \operatorname{Re} z} \qquad (z \in l).$$

This form of notation may also be retained in the general case, if one regards the operator R_z as defined not on all of B, but only on some of its subspace (depending on z).

2°. In a similar way one may consider the equation

$$\sum_{k=0}^{n} A_k \frac{d^k u}{dt^k} = 0; \qquad \left.\frac{d^k u}{dt^k}\right|_{t=0} = 0 \quad (k = 0, \dots, n-1). \tag{5}$$

Here A_k $(k = 0, 1, \dots, n)$ are closed linear operators in B; the solution is understood in the weak sense. In this case, in condition A inequality (2) should be replaced by the inequality $(u \in \bigcap D(A_k),\ z \in l)$

$$\left\| \sum_{k=0}^{n} z^k A_k u \right\| \geqslant Ce^{-\sigma \operatorname{Re} z} \|u\|. \tag{6}$$

Theorem 2. *If condition A is satisfied with (2) replaced by (6) and $0 < \sigma < T$, then the solution of the Cauchy problem is identically equal to zero on $[0, T - \sigma]$.*

Obviously, the corollaries of Theorem 1 are valid, with the corresponding changes, also for equation (5).

In particular, for the Cauchy problem

$$\frac{d^2 u}{dt^2} = Au, \qquad u(0) = u'(0) = 0 \tag{7}$$

the condition ensuring local uniqueness has the form (after the substitution $z^2 = \zeta$)

$$\|Au - \zeta u\| \geqslant C_\sigma e^{-\sigma |\zeta|^{1/2}} \|u\|$$

$(u \in D_A,\ \sigma$ is an arbitrary positive number), where ζ runs along one of the branches of an arbitrary parabola with focus at the point $\zeta = 0$ and axis,

distinct from the negative real half-axis; this parabola, in particular, may degenerate into a ray emanating from the origin and also not coinciding with the negative real half-axis.

3°. We now suppose that B is a Hilbert space. Denote by $W_0(A)$ the set of values of the quadratic form of the operator A on the unit sphere:

$$W_0(A) = \{z : z = (Au, u), \quad u \in D_A, \quad \|u\| = 1\};$$

let $W(A)$ be the closure of $W_0(A)$. It is known (see (⁵), pp. 130-132) that $W(A)$ is a closed convex set in the complex plane.

Theorem 3. *If $W(A)$ coincides neither with the whole plane nor with any right half-plane $\operatorname{Re} z \geq a$, then for the Cauchy problem (1) local uniqueness holds.*

Indeed, in this case, by virtue of the convexity of $W(A)$, it is possible to draw a positive ray l lying at a positive distance d from $W(A)$. Then for $z \in l$, $u \in D_A$, $\|u\| = 1$, the inequality

$$|(Au - zu, u)| = |(Au, u) - z| \geq d > 0$$

is satisfied, since $(Au, u) \in W(A)$. Hence we obtain that for all $u \in D_A$, $z \in l$,

$$\|Au - zu\| \geq d\|u\|,$$

and this, by Corollary 2 of Theorem 1, ensures local uniqueness for the Cauchy problem (1).

Let us note that for the Cauchy problem (7) the exceptional case is the unique one in which $W(A)$ coincides with the whole plane.

Example. Let A_0 be a maximal Hermitian operator and $A = A_0^*$. Then for the Cauchy problem (1) local uniqueness holds.

Indeed, since one of the defect numbers of the operator A_0 is equal to zero, in Neumann' s formula (see (⁶), p. 132)

$$\operatorname{Im}(Au, u) \equiv \operatorname{Im}(A_0^* u, u) = \|u^+\|^2 - \|u^-\|^2$$

(where $u^- \in \mathfrak{N}_i$, $u^+ \in \mathfrak{N}_{-i}$, $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$ are the defect subspaces of the operator A_0) one of the terms $\|u^+\|^2$ or $\|u^-\|^2$ disappears; consequently, $\operatorname{Im}(Au, u)$ has one and the same sign for all $u \in D_A$, i.e. $W(A)$ is entirely contained either in the upper or in the lower half-plane, and this ensures local uniqueness for the Cauchy problem (1). Let us note that an analogous result is also valid for the operator $A = \lambda A_0$, where λ is an arbitrary non-purely imaginary number ($\operatorname{Re} \lambda \neq 0$).

4°. Let B still be a Hilbert space, and let the operator A satisfy the condition ($u \in D_A$)

$$\operatorname{Re}(Au, u) \leq c(u, u). \tag{8}$$

For $x > c$, $u \in D_A$,

$$\|Au - xu\|^2 = \|Au - cu\|^2 - 2(x - c)\operatorname{Re}(Au - cu, u) + (x - c)^2\|u\|^2 \geq$$
$$\geq \|Au - cu\|^2 + (x - c)^2\|u\|^2,$$

whence for $z = x + iy = c + re^{i\varphi}$, $|\varphi| < \pi/2$, and for arbitrary $\varepsilon > 0$ we easily obtain

$$\|Au - zu\|^2 \geq \frac{\varepsilon}{1+\varepsilon}\|Au - cu\|^2 + \varepsilon(x - c)^2\left(\frac{1}{1+\varepsilon} - \operatorname{tg}^2\varphi\right)\|u\|^2. \tag{9}$$

Further,

$$-\operatorname{Re}(Au - zu, u) = -\operatorname{Re}(Au, u) + x(u, u) \geq (x - c)\|u\|^2,$$

whence

$$\|Au - zu\|^2 \geq (x - c)^2\|u\|^2. \tag{10}$$

Multiplying inequality (9) by λ, $0 \leq \lambda \leq 1$, and inequality (10) by $1 - \lambda$ and adding, we obtain ($k = |\operatorname{tg}\varphi|$)

$$\|Au - zu\|^2 \geq \frac{\lambda\varepsilon}{1+\varepsilon}\|Au - cu\|^2 + \left(\frac{\lambda\varepsilon}{1+\varepsilon} - \lambda\varepsilon k^2 + 1 - \lambda\right)(x - c)^2\|u\|^2.$$

Choosing λ and ε in an optimal way, we arrive at the inequality

$$\|Au - zu\|^2 \geq p\|Au\|^2 + C_p|z|^2\|u\|^2, \tag{11}$$

valid for $|z| \geq r_0$ and $p < P(k)$, where

$$P(k) = \begin{cases} 1 - k^2, & \text{for } 0 \leq k \leq k_0 = \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1}{k(k+2)}, & \text{for } k \geq k_0. \end{cases}$$

For the operator $A = e^{i\alpha}A_0$, $|\alpha| < \pi$, where A_0 satisfies (8), inequality (11) remains valid for $z = re^{i\varphi}$ with $|\varphi - \alpha| < \pi/2$ and $k = |\operatorname{tg}(\varphi - \alpha)|$. Hence it is easy to obtain that there always exists a positive ray $z = re^{i\varphi}$ ($|\varphi| < \pi/2$), along which inequality (11) holds for arbitrary $p < P_1(\alpha)$, where

$$P_1(\alpha) = \begin{cases} 1, & \text{for } |\alpha| \leq \pi/2, \\ 1 - \operatorname{ctg}^2\alpha, & \text{for } \pi/2 \leq |\alpha| \leq \alpha_0 = \pi - \operatorname{arc\,ctg}\dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1}{|\operatorname{ctg}\alpha|(|\operatorname{ctg}\alpha| + 2)}, & \text{for } \alpha_0 \leq |\alpha| < \pi \end{cases}$$

(it is enough to put $\varphi = \alpha$ for $|\alpha| < \pi/2$, and $\varphi = \pm(\pi/2 - \varepsilon)$ for $\pi/2 \leq |\alpha| < \pi$). The existence of inequality (11) makes it possible to establish the following result:

Theorem 4. *If the operator A has the form $A = e^{i\alpha}A_0 + A_1$, where $|\alpha| < \pi$, A_0 satisfies (8), and A_1 is subordinate to A_0: for $u \in D(A_0) \subset D(A_1)$*

$$\|A_1 u\|^2 \le p\|A_0 u\|^2 + C\|u\|^2, \tag{12}$$

with $p < P_1(\alpha)$*, then local uniqueness holds for the Cauchy problem (1).*

If the operator A_0 satisfies the condition

$$c_1(u,u) \le \mathrm{Re}(A_0 u, u) \le c_2(u,u),$$

then uniqueness is preserved if in inequality (12) $p < 1$ (and α is arbitrary). The same circumstance also holds for a Hermitian operator A_0.

Kyiv State University
named after T. G. Shevchenko

Received
17 VII 1962

REFERENCES

1. Yu. I. Lyubich, DAN, **130**, No. 5, 969 (1960).
2. Yu. I. Lyubich, UMN, **16**, issue 5, 181 (1961).
3. I. M. Gelfand, G. E. Shilov, *Generalized functions and operations on them*, Moscow, 1958.
4. G. E. Shilov, *Mathematical Analysis*, Moscow, 1960.
5. M. N. Stone, *Linear Transformations in Hilbert Space*, N.-Y., 1932.
6. M. A. Naimark, *Linear Differential Operators*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.