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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

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## ON SOME EXTREMAL PROPERTIES OF POLYNOMIALS

*(Presented by Academician S. N. Bernstein on 12 VII 1963)*

In this note an elementary proof is given of one inequality concerning trigonometric polynomials, from which, as a consequence, a number of extremal properties of polynomials are obtained.

**Theorem.** Let  $S(\theta)$  be an arbitrary trigonometric polynomial of degree  $n$ . For arbitrary real  $\theta, \varphi, \tau$  the inequality

$$\begin{aligned} & |e^{i\varphi} S(\theta + i\tau) + e^{-i\varphi} S(\theta - i\tau)| \leq \\ & \leq 2\sqrt{\cos^2 \varphi + \operatorname{sh}^2 n\tau} \max |S(\theta)|, \quad 0 \leq \theta < 2\pi. \end{aligned} \quad (1)$$

For the proof of this theorem we shall use the following lemma.

**Lemma.** If  $P(z)$  is an algebraic polynomial of degree  $m$ , all of whose zeros are situated in the circular domain  $|z| > 1$ , then the zeros of every equation of the form

$$P(\alpha z) + \eta \bar{\alpha}^m P(\bar{\alpha}^{-1} z) = 0, \quad |\eta| = 1, \quad |\alpha| \neq 1, \quad (2)$$

are situated in the same domain.

To prove this lemma, note that from  $|z| \leq 1$ ,  $|\alpha| > 1$ , and  $|\omega| > 1$  it follows that

$$|\alpha z - \omega| < |z - \bar{\alpha}\omega|,$$

and, consequently, if  $P(z)$  has the form

$$P(z) = \prod_{\nu=1}^m (z - \omega_{\nu}), \quad |\omega_{\nu}| > 1,$$

and if  $|\alpha| > 1$ , then for  $|z| \leq 1$  we shall have

$$|P(\alpha z)| = \prod_{\nu=1}^m |\alpha z - \omega_{\nu}| < \prod_{\nu=1}^m |z - \bar{\alpha}\omega_{\nu}| = |\bar{\alpha}^m P(\bar{\alpha}^{-1} z)|.$$

From this inequality it is clear that in the case  $|\alpha| > 1$  equation (2) has no roots in the unit disk  $|z| \leq 1$ . The case  $|\alpha| < 1$  can be considered in the same way or reduced to the preceding one.

We return to the proof of the theorem formulated at the beginning. For this purpose we represent  $S(\theta)$  in the form

$$S(\theta) = e^{-ni\theta}Q(e^{i\theta}),$$

where  $Q = Q(z)$  is an algebraic polynomial of degree  $m \leq 2n$ . Without loss of generality one may suppose that  $m = 2n$ , since this can always be achieved by small changes in the coefficients of  $S(\theta)$ . In addition, we shall assume that  $\tau \neq 0$ . Put

$$\max_{\theta} |S(\theta)| = L \quad (0 \leq \theta < 2\pi);$$

then

$$|Q(z)| \leq L \quad \text{for } |z| = 1. \quad (3)$$

Obviously, we shall have

$$2\sqrt{\cos^2 \varphi + \text{sh}^2 n\tau} = |e^{i\varphi} \alpha^{-n} + e^{-i\varphi} \alpha^n|,$$

where  $\alpha = e^{-\tau}$ , and inequality (1) can be written in the form

$$|e^{i\varphi} \alpha^{-n} Q(\alpha z) + e^{-i\varphi} \alpha^n Q(\alpha^{-1} z)| \leq L |e^{i\varphi} \alpha^{-n} + e^{-i\varphi} \alpha^n|, \quad |z| = 1. \quad (4)$$

Suppose that for some real  $\theta = \theta_0$  inequality (1) is false. Then, for  $z = e^{i\theta_0}$ , inequality (4) will also be violated. Conse-

Consequently, we shall have an equality of the form

$$e^{i\varphi} \alpha^{-n} P(\alpha \xi) + e^{-i\varphi} \alpha^n P(\alpha^{-1} \xi) = 0, \quad \xi = e^{i\theta_0}, \quad (5)$$

where  $P(z) = Q(z) - \lambda$ , and  $\lambda$  is some complex number with modulus  $> L$ . From (3) and from the maximum-modulus principle it follows that all the zeros of the polynomial  $P(z) = Q(z) - \lambda$  ( $|\lambda| > L$ ) lie outside the unit circle  $|z| \leq 1$ . On the basis of the lemma already proved, the zeros of the equation

$$P(\alpha z) + e^{-2i\varphi} \alpha^{2n} P(\alpha^{-1} z) = 0$$

must also be situated in the region  $|z| > 1$ , which, however, contradicts equality (5). This proves the theorem. We note that an analogous method of proof was used in [6].

Let us note the following special cases of inequality (1):

$$|S(\theta + i\tau) - S(\theta - i\tau)| \leq 2L \text{ sh } n|\tau|, \quad (6)$$

$$|S(\theta + i\tau) + S(\theta - i\tau)| \leq 2L \text{ ch } n\tau. \quad (7)$$

From (6) and (7) follows the well-known inequality

$$|S(\theta + i\tau)| \leq L e^{n|\tau|}. \quad (8)$$

In the case when  $S(\theta)$  is a trigonometric polynomial with real coefficients, from (1) one obtains the inequality

$$|S(\theta + i\tau)| \leq L \sqrt{\cos^2 \psi + \operatorname{sh}^2 n\tau}, \quad \psi = \arg S(\theta + i\tau). \quad (9)$$

In particular, we have:

$$|S(\theta + i\tau)| \leq L \operatorname{ch} n\tau \quad (S(\theta) \text{ with real coefficients}). \quad (10)$$

Inequality (6) is analogous to the following inequality:

$$|S(\theta + h) - S(\theta - h)| \leq 2 \sin nh \cdot \max_{\theta} |S(\theta)|, \quad 0 < h \leq \frac{\pi}{2n},$$

which was proved by S. N. Bernstein in general form for integral functions of finite degree in [4]. This inequality was generalized by B. Ya. Levin [5].

We should indicate several assertions which are consequences of the inequalities proved above.

A. From (6), by passing to the limit as  $\tau \rightarrow 0$ , we obtain S. N. Bernstein's inequality

$$|S'(\theta)| \leq nL.$$

B. From (6) (and also from (7)), by passing to the limit as  $\tau \rightarrow \infty$ , we easily obtain the inequality (see [1], p. 28)

$$\max_{0 \leq \theta \leq 2\pi} \left| \sum_{\nu=0}^n (a_{\nu} \cos \nu\theta + b_{\nu} \sin \nu\theta) \right| \geq \max_{0 \leq \theta \leq 2\pi} |a_n \cos n\theta + b_n \sin n\theta|. \quad (11)$$

C. Let  $P(x)$  be an arbitrary algebraic polynomial of degree  $n$ . Then  $S(\theta) = P(\cos \theta)$  is an even trigonometric polynomial of degree  $n$ , and inequality (7) for  $\theta = 0$  takes the form

$$|P(\operatorname{ch} \tau)| \leq L \operatorname{ch} n\tau, \quad (12)$$

where

$$L = \max_{0 \leq \theta \leq 2\pi} |S(\theta)| = \max_{-1 \leq x \leq 1} |P(x)|.$$

Inequality (12) was first proved by S. N. Bernstein ([2], p. 21).

Let  $z$  be an arbitrary point of the complex plane not belonging to the interval  $[-1, 1]$ , and let  $R$  be the semisum of the axes of the ellipse passing through  $z$  and having foci at the points  $-1$  and  $+1$ . Inequality (8), in the case where  $S(\theta)$  is the polynomial  $P(\cos \theta)$ , can be written in the form

$$|P(z)| \leq LR^n. \quad (13)$$

This inequality was first given by S. N. Bernstein <sup>(2,3)</sup>.

We note that in the case where the polynomial  $P(x)$  has real coefficients, instead of (13), as is evident from (10), one can write the more precise inequality (see <sup>(1)</sup>, note on p. 75):

$$|P(z)| \leq \frac{L}{2}(R^n + R^{-n}). \quad (14)$$

D. Let  $S(\theta)$  be a real trigonometric polynomial,  $\tau$  a fixed real number, and  $\xi = \operatorname{Re} S(\theta + i\tau)$ ,  $\eta = \operatorname{Im} S(\theta + i\tau)$ . Then inequality (1) can be transformed into the form

$$|\xi \cos \varphi - \eta \sin \varphi| \leq L \sqrt{\cos^2 \varphi + \operatorname{sh}^2 n\tau}, \quad 0 \leq \varphi < 2\pi. \quad (15)$$

It is easily verified that the intersection of all the strips (15) coincides with the ellipse

$$\frac{\xi^2}{\operatorname{ch}^2 n\tau} + \frac{\eta^2}{\operatorname{sh}^2 n\tau} \leq L^2. \quad (16)$$

Thus, the point  $S(\theta + i\tau)$  must lie in the ellipse (16).

E. If item D is applied to the polynomial  $S(\theta) = P(\cos \theta)$ , we obtain the following assertion, more general than inequality (14):

Under the mapping  $\xi + i\eta = P(x + iy)$ , where  $P(z)$  is a real algebraic polynomial of degree  $n$ , the image of every ellipse

$$\frac{x^2}{\operatorname{ch}^2 \tau} + \frac{y^2}{\operatorname{sh}^2 \tau} \leq 1$$

( $\tau > 0$ ) belongs to the ellipse

$$\frac{\xi^2}{\operatorname{ch}^2 n\tau} + \frac{\eta^2}{\operatorname{sh}^2 n\tau} \leq L^2, \quad L = \max_{-1 \leq x \leq 1} |P(x)|.$$

Here the extremal case is the one in which  $P(z)$  coincides with the Chebyshev polynomial.

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*Note: Figure translations are in progress. See original paper for figures.*

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