Reports of the Academy of Sciences of the USSR

1963

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1963. Volume 153, No. 1

MATHEMATICS

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ON THE TOPOLOGICAL CHARACTERIZATION OF UNIFORM PROPERTIES OF METRIC SPACES

(Presented by Academician P. S. Aleksandrov, 24 V 1963)

Consider the class $\{P_\alpha\}$ of all metric spaces P_α. Each space P_α has a unique bicompact extension* uP_α, generating the natural proximity** of the space P_α. Thus to each space P_α there is associated a topological space

$$N_\alpha = uP_\alpha \setminus P_\alpha,$$

which we shall call the **remainder** of the space P_α. Yu. M. Smirnov posed the question of characterizing (and also of the possibility of such a characterization of) uniform properties*** of spaces P_α by topological properties of their remainders N_α. The first nontrivial example of this was given in (^{5}): the uniform property of total boundedness of the spaces P_α is equivalent to the topological property of hereditary normality of their remainders N_α. In the same place, as a "trivial" consequence of one theorem of Yu. M. Smirnov,**** it was asserted that the uniform property of completeness of the space P_α is equivalent to the property that its remainder does not satisfy the first axiom of countability. Here we prove this assertion, which at first seemed simple (see Theorem 1), and even in a strengthened form. In addition, we give several uniform properties which cannot be characterized in the indicated way (see Theorem 2). Let us also note that in the class of all proximity spaces such a "proximity" property*****, as the property of metrizability, cannot be characterized by any topological property of the remainder. This follows from a result of Yu. M. Smirnov, which asserts that every completely regular space can be the remainder for some pseudocompact nonmetrizable space.******

Theorem 1. Let P be a metric space, $N = uP \setminus P$. The following assertions are equivalent to one another:

a) every nonempty bicompact subset of the remainder N of type G_δ in N has cardinality not less than the cardinality of the hypercontinuum*******;

b) no point of the remainder N has type G_δ in it;

c) the remainder N does not satisfy the first axiom of countability;

d) the space P is complete.

Remark 1. Of course, assertion a) may be replaced by any assertion intermediate between it and assertion c). For example, every nonempty bicompact subset of the remainder N that is countable the remainder in N has uncountable character. Here the countability of the character (and also of the pseudocharacter) is essential*: every point of the remainder of the full space of all natural numbers has continuum character.

Remark 2. For arbitrary proximity spaces this theorem is not true even in the case of local bicompactness: any bicompactum N without the first axiom of countability, according to Yu. M. Smirnov, can be the remainder of a pseudocompact (nonmetrizable) space P. By virtue of pseudocompactness, this space is totally bounded** in any proximity compatible with the topology, in particular in the proximity generated by the extension $uP = P \cup N$. Consequently, the bicompactum uP itself is a completion*** of the space P.

Preliminary lemmas. Let aT be a bicompact extension of the space T. We shall call a point x of the remainder $aT \setminus T$ **attainable** if it is a limit point for some countable subset K of the space T.

Lemma 1. Let aT be a bicompact extension of a finally compact space T; every bicompact subset Φ of the remainder $aT \setminus T$ of type G_δ (in the remainder) has countable character in aT.

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* See (3), p. 557, Theorem 10.

** Subsets A and B of a metric space P with metric ρ are considered close in it if $\rho(A, B) = 0$.

*** That is, properties preserved under uniformly continuous (in both directions) homeomorphisms.

**** See (3), p. 287, Theorems 7 and 7′. Yu. M. Smirnov proved that the property of completeness of the space P_α is equivalent to the property that the bicompactum uP_α not satisfy the first axiom of countability at any point of the remainder (in other words, all points of the remainder have uncountable character in the bicompactum uP_α).

***** That is, a property preserved under mappings that take close sets to close sets and distant sets to distant sets.

****** Nonmetrizable even in the topological sense; see (4), p. 299.

******* That is, the cardinality of the set of all subsets of the line.

Lemma 2. For any weakly paracompact**** space T and for any of its regular extensions aT, the set of all attainable points of the remainder $aT \setminus T$ is dense in $aT \setminus T$.

Lemma 3. If a bicompact extension aT of a space T satisfies the assertion of Lemma 2, then every nonempty subset of the remainder $aT \setminus T$ having countable character (in the remainder!) contains attainable points.

Proof of the theorem. It is not difficult to see that each assertion of Theorem 1 is stronger than the following one, and property b) is stronger than property c) by virtue of the cited theorem of Yu. M. Smirnov. Let now P be complete, and let a nonempty bicompactum Φ lie in the remainder N and have type G_δ in it.

1. The space P has a countable base. Then, by Lemma 1, the bicompactum Φ has countable character in uP, i.e., there exists such a countable system of neighborhoods $\{O_k\Phi\}$ of the bicompactum Φ that in any of its neighborhoods $O\Phi$ there is contained at least one neighborhood $O_k\Phi$. We may assume that $\overline{O_{k+1}\Phi}^u \subset O_k\Phi$***** for every k. Since $\Phi \neq \emptyset$******, all neighborhoods $O_k\Phi$ are nonempty, and with them the intersections $P \cap O_k\Phi$ are also nonempty. Let
$$p_k \in P \cap \overline{O_k\Phi}^u.$$
Since
$$\bigcap_k \overline{O_k\Phi}^u = \Phi \subset N,$$
all limit points of the sequence $\{p_k\}$ are contained in Φ. Hence, from the completeness of the space P we see that no subsequence of the sequence $\{p_k\}$ is fundamental. Then, by one lemma of Yu. M. Smirnov*******, there exist a subsequence $C = \{p_{k_i}\}$ and a positive number ε such that all pairwise distances between distinct points of this sequence C are greater than ε. According to the theory

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* The character of a set M in a space P is the least of all such cardinalities τ that are the cardinalities of bases of the space P around the set M. The pseudocharacter of a set M is the least of all such cardinalities that are the cardinalities of systems of open sets whose intersection is the set M.

** See (3), Theorem 12. A space P is totally bounded if every open cover of each uniform structure compatible with the proximity of this space can be prolonged to an open cover of the bicompactum uP.

*** The completion of a proximity space P is the largest of all such subspaces of the bicompactum uP to each of which every cover by sets open in P of a uniform structure compatible with P can be prolonged to an open cover.

**** A space is called weakly paracompact if into each of its open covers one can inscribe a point-finite (i.e., each point belongs to only a finite number of elements of the cover ω) open cover ω.

***** By $\overline{M}^u$ we denote the closure of the set M in the bicompactum uP.

****** By $\emptyset$ we denote the empty set.

******* See (3), p. 286.

Yu. M. Smirnov*, we have: $\overline{C}^u = uC = \beta C$**. Further we have $\beta C \setminus C = \overline{C}^u \setminus C \subseteq \Phi$. Since the remainder $\beta C \setminus C$ has the cardinality of the continuum, Φ also has cardinality at least that of the continuum.

p. 2. The bicompactum Φ contains at least one accessible point x, i.e. $x \in \overline{K}^u$, where K is countable and lies in P. But $\overline{K}^u = \overline{K} = uK$***, and $\overline{K}$ is a complete subspace of the subspace P and has a countable base. Hence the intersection $\Phi \cap \overline{K}^u$ is a nonempty bicompactum of type G_δ in the remainder $\overline{K}^u \setminus \overline{K}$, equal to $uK \setminus \overline{K}$. Hence, by p. 1, we conclude that the set $\Phi \cap \overline{K}^u$, and therefore also the whole bicompactum Φ, has cardinality not less than the cardinality of the continuum.

p. 3. The space P is locally bicompact. In this case, since the remainder N is bicompact, the bicompactum Φ has countable character in it. According to Lemmas 2 and 3, the bicompactum Φ contains an accessible point, and everything reduces to p. 2.

p. 4. **The general case.** Since Φ has type G_δ in N, we have $N \setminus \Phi = \bigcup\limits_k H_k$, where k runs through the natural numbers, and all the sets H_k are closed in N. Since $\Phi \cap \overline{H}_k^u = \emptyset$, there exist sets G_k open in uP such that $\overline{H}_k^u \subseteq \overline{G}_k^u \subseteq G_{k+1} \subseteq uP \setminus \Phi$ for every k. Let $\Pi = \bigcup\limits_k G_k = \bigcup\limits_k \overline{G}_k^u$, and $P_0 = P \setminus \Pi$. It is clear that $\Phi = N \setminus \Pi = uP \setminus P_0 \setminus \Pi$. If $\Phi \cap \overline{P}_0^u = \emptyset$, then $\Phi = uP \setminus \bigcup\limits_k \overline{G}_k^u \setminus \overline{P}_0^u$, and hence the bicompactum Φ has type G_δ in uP. Since uP is bicompact, Φ has countable character in uP. Therefore, by Lemmas 2 and 3, everything again reduces to p. 2.

Finally, suppose $\Phi \cap \overline{P}_0^u \neq \emptyset$. Since P_0 is closed in P, P_0 is complete. We also note that $\overline{P}_0^u = uP_0$, $\Phi \cap P_0 = \emptyset$, and $\overline{P}_0^u \setminus \overline{P}_0 = uP_0 \setminus P_0 \subseteq uP \setminus \Pi \setminus P_0 = \Phi$. Therefore the set P_0 is locally bicompact. Moreover, the bicompactum $\Phi \cap \overline{P}_0^u$ is nonempty and has type G_δ in the remainder $uP_0 \setminus P_0$. Thus, by p. 3, we again see that the intersection $\Phi \cap \overline{P}_0^u$, and together with it the bicompactum Φ, have cardinality not less than the cardinality of the continuum. The theorem is proved.

Corollary. All points of the completion cP of the metrizable space P which do not belong to P itself are precisely those points of the remainder N which have countable character in N, or, equivalently, all those points of the remainder which have type G_δ in it.

Indeed, it is known that $ucP = uP$ and that for regular extensions of the space P the character of any point x from P with respect to the extension is equal to the character of the point x with respect to the space P****. This, together with Theorem 1, gives both assertions of the corollary.

Proof of the lemmas. 1. By bicompactness of the extension aT, it is enough to prove that, under the conditions of the lemma, the bicompactum has type G_δ in aT. This latter assertion is true even in the case of regularity of the extension aT. Indeed, let $N \setminus \Phi = \bigcup\limits_k H_k$, where all H_k are closed in $N = aT \setminus T$.

For every point x of the set $aT \setminus \Phi$ open in aT there exists a neighborhood Ox such that $\Phi \cap \overline{Ox}^a = \emptyset$. From the open covering $\omega = \{Ox : x \in T\}$ choose a countable subcovering $\{Ox_i\}$ of the finally compact space T. It is clear that $aT \setminus \Phi = \bigcup\limits_k \overline{H}^a_k \cup \bigcup\limits_i \overline{Ox}^a_i$, since $\Phi \cap \overline{H}^a_k = \emptyset$. The lemma is proved.

2. Take an arbitrary nonempty open set H' of the remainder $N = aT \setminus T$, where T is weakly paracompact. Choose for it open in aT sets H and G so that $N \cap H = H'$, $\overline{G}^a \subset H$, and $N \cap G \neq \emptyset$. Suppose that in H' there are no approachable points. Then there are none in $N \cap \overline{G}^a$ either. Hence every countable set K lying in $T \cap \overline{G}^a$ has a limit point in it, i.e. the set $T \cap \overline{G}^a$, closed in T, is (countably-)compact. At the same time $T \cap \overline{G}^a$ is weakly paracompact and therefore, by a theorem of B. T. Levshenko (^{1}), Theorem 1, bicompact. It is easy to see that $\overline{T \cap G^a} = \overline{G}^a$. But then $\overline{T \cap G^{a\,a}} = \overline{G}^a$, and, by the bicompactness of the set $T \cap \overline{G}^a$, we find that $T \cap \overline{G}^a = \overline{G}^a$, which contradicts the choice of the set G. The lemma may be regarded as proved.

3. Let, under the hypotheses of Lemma 3, Φ be a nonempty set of the accretion $N = aT \setminus T$ of countable character in N. Then there exist open sets H_k of this accretion such that for every neighborhood $O\Phi$ of the set Φ there is some set H_k satisfying the inclusions $\Phi \subset H_k \subset O\Phi$. By assumption there exist approachable points x_i such that

$$x_i \in \bigcap_{j \leq i} H_j.$$

Let K_i be such a countable subset of the space T that $x_i \in \overline{K_i}^a$. The sequence

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* See (2), Ch. 2.
** By βC we denote the Čech bicompact extension of the space C.
*** If $M \subset P$, then by $\overline{M}$ we denote the closure of the set M in P.
**** See Theorems 1 and 5 and Lemma 11 in (3).

$C = \{x_i\}$ has a limit point x in Φ. Indeed, if this were not so, then in the neighborhood $O\Phi = aT \setminus \overline{C}^a$ there would be no point x_i, which, in view of the special choice of the sets H_j and the points x_i, is impossible. It is easy to see that $x \in \overline{K}^a$, where $K = \bigcup K_i$. Consequently, x is an approachable point. The lemma is proved.

Theorem 2. *If a metric space P is representable as the sum of such a sequence of compact sets Φ_k, whose diameters tend to zero, that, beginning with some k_0, all distances $\rho(\Phi_{k'}, \Phi_{k''})$ are bounded below by a positive number, then its accretion $uP \setminus P$ is homeomorphic to the accretion of the sequence of natural numbers.*

Proof. Let $\rho(\Phi_{k'}, \Phi_{k''}) \geq \delta > 0$ for all possible k' and k, $k' \neq k$, exceeding the number k_0, and let $x_k \in \Phi_k$ for $k > k_0$. The sequence $C = \{x_k\}$ is proximally homeomorphic to the sequence of natural numbers and has no limit points in P. Hence $\beta C = uC = \overline{C}^u$. Therefore it is enough to prove the equality $\overline{C}^u \setminus C = uP \setminus P$, and even only the inclusion $uP \setminus P \subseteq \overline{C}^u \setminus C$. Suppose that there exists a point x of the difference $uP \setminus P \setminus \overline{C}^u$. Since it is far from the bicompactum $\overline{C}^u$, there exist disjoint neighborhoods Ox of the point x and a proximal neighborhood H of the bicompactum $\overline{C}^u$. The intersection $H \cap P$ is a proximal neighborhood of the set C, and therefore there is a number $\varepsilon > 0$ such that the ε-neighborhood $O_\varepsilon C$ is contained in H. By the condition of the theorem, beginning with some $k_1 \geq k_0$, all compact sets Φ_k lie in $O_\varepsilon C$. But then the difference $Ox \setminus \bigcup_{k<k_1} \Phi_k$ contains no point of P, being an open set in uP, which is impossible. The theorem is proved.

Corollary. *The following uniform properties of metric spaces cannot be characterized by topological properties of accretions: the property of being n-dimensional in the sense of the dimension δd of Yu. M. Smirnov, and the property "every continuous function is uniformly continuous."*

Indeed, taking in n-dimensional Euclidean space, on the axis Ox, the sequence of natural numbers x_k and the sum P of the neighborhoods $O_{1/k}x_k$, we see that the sequence $\{x_k\}$ and the sum P have homeomorphic accretions, although each of the indicated properties holds on one space and does not hold on the other.

In conclusion I take the opportunity to express my deep gratitude to Yu. M. Smirnov for his attention to my work.

Received
23 V 1963

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