ON METRIC SPACES AND CONTINUOUS MAPPINGS ASSOCIATED WITH THEM

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Abstract

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ON METRIC SPACES AND CONTINUOUS MAPPINGS ASSOCIATED WITH THEM

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The present note is close in its methods to the work (^{6}), in which the concept of a projection spectrum was given in full generality (^{1}). For all questions connected with this concept, see ($^{6-9}$).

Let us recall some concepts that will occur in this note. A set* of coverings $\mathfrak{A} = \{\omega\}$ of a space X (closed or open) will be called **weakly refining** if, for every neighborhood Ox_0 of an arbitrary point $x_0 \in X$, the star $\Gamma_\omega x_0$ of this point with respect to some covering $\omega \in \mathfrak{A}$ is contained in Ox_0. A set of coverings $\mathfrak{A} = \{\omega\}$ will be called **complete** if, for every centered system $\xi = \{A_\omega\}$ of sets, taken one from each covering $\omega \in \mathfrak{A}$, necessarily

$$\bigcap [A_\omega] \neq \Lambda.$$

Theorem 1. *The following assertions are equivalent to one another:*

a) *The space X is metrizable by a complete metric ρ.*

b) *In the space X there exists a countable complete sequence of locally finite open coverings ω_i, which together form a base.*

c) *In the space X there exists a countable complete weakly refining sequence of locally finite coverings ω_i such that ω_i is star-refined into the covering ω_{i-1} for every i.*

d) *In the space X there exists a countable weakly refining complete sequence of decompositions $\Sigma = \{\alpha_i\}$.*

e) *The space X is the space of a countable projection spectrum*

$$S = \{|\alpha_i|,\ \tilde{\omega}_i^{i+1}\}.$$

f) *The space X is a perfect irreducible image of some complete metric space of dimension $\dim \Xi = 0$ (lying in the Baire space B^τ of weight τ, equal to the weight of X).*

We outline the proof of this theorem.

1°. d) → e). If in the space X there is a weakly refining complete countable sequence of decompositions $\Sigma = \{\alpha_i\}$, then to this sequence Σ there corresponds a countable spectrum

$$S = \{|\alpha_i|,\ \tilde{\omega}_i^{i+1}\}$$

of the nerves of these decompositions with the natural projections

$$\tilde{\omega}_i^{i+1} : |\alpha_{i+1}| \to |\alpha_i|$$

for $\alpha_{i+1} \to \alpha_i$, following from the order established on the set of decompositions Σ. Since the sequence Σ is complete and weakly refining, the space $\widetilde{S}$ of this spectrum is homeomorphic to the space X.

2°. e) → f). Along with the spectrum

$$S = \{|\alpha_i|,\ \tilde{\omega}_i^{i+1}\}$$

consider the spectrum

$$\dot{S} = \{\dot{\alpha}_i,\ \tilde{\omega}_i^{i+1}\},$$

which is a complete relaxation of the spectrum S (6,7). Here $\dot{\alpha}_i$ is the zero-dimensional complex consisting of all vertices of the complex $|\alpha_i|$. The space $\dot{S}$ of the spectrum $\dot{S}$ is a zero-dimensional (in the sense of dim) metrizable space with a complete metric. Indeed, in the space $\dot{S}$ the spectral coverings

$$\varphi_i = \{\Phi_{e_i}\}, \qquad \Phi_{e_i} = \mathcal{E}(\xi = \{e'_j\},\ e'_i = e_i)$$

are disjoint, and the system $\sigma = \bigcup \varphi_i$ forms an open base in $\widetilde{S}$. Hence it follows that $\widetilde{S}$ is metrizable and has dimension $\dim \widetilde{S} = 0$. The completeness of the system of coverings $\{\varphi_i\}$ follows from

$$\widetilde{S} = \lim_{\leftarrow} \{|\alpha_i|, \mathfrak{S}_i^{i+1}\}.$$

Hence, by A. Arhangel' skii' s theorem (2), it follows that the metric space $\widetilde{S}$ has a complete metric. The spectral mapping $\pi_X : \widetilde{S} \to S = X$ (see (6,7)), generated by the natural passage from the spectrum $\dot{S}$ to the spectrum S, is perfect and irreducible, whereby assertion e) is proved.

3°. e) ⇒ a). The space X, as a perfect image of the metric space $\widetilde{S}$, is metrizable by Stone' s well-known theorem (11); by the theorem proved by me in (10), the

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* When we speak of a set of coverings $\mathfrak{A} = \{\omega\}$ of a space, we do not suppose that any order has been introduced in $\mathfrak{A}$. By a sequence of coverings $\mathfrak{A} = \{\omega\}$ one should understand a set of coverings endowed with a natural partial order, in which they are directed, i.e., for any two $\omega_1, \omega_2 \in \mathfrak{A}$ there is an $\omega_3 \succ \omega_1$ and $\omega_3 \succ \omega_2$.

space X, as the image of a complete metric space, is complete in the sense of Čech and, being metrizable, admits a complete metric.

4°. The equivalence of a), b), c) is proved without difficulty (for example, relying on A. Arhangel' skii' s theorem (^{2})).

5°. c) ⇒ d). Let $\mathfrak{A} = \{\omega_i\}$ be a countable, weakly refining complete set of such open coverings that ω_{i+1} is star-refined into the covering ω_i for every i. Consider such a sequence of partitions α_i that, for each i, the partition α_i is inscribed both in the open covering ω_i and in the partition α_{i-1}. The set $\Sigma = \{\alpha_i\}$ is a countable complete weakly refining sequence of partitions of the space X.

The proof of Theorem 1 is complete.

Corollary of Theorem 1. *In order that a space be metrizable, it is necessary and sufficient that any one of the following conditions be fulfilled:*

a) *In the space X there exists a weakly refining countable sequence of partitions $\Sigma = \{\alpha_i\}$.*

b) *The space X is contained in the space $\widetilde{S}$ of a countable projection spectrum*

$$S = \{|\alpha_i|, \mathfrak{S}_i^{i+1}\}.$$

c) *The space X is a perfect irreducible image of some metric space of dimension $\dim \Xi = 0$.*

For what follows we shall need the following

Definition. A spectrum $S = \{|\alpha|, \mathfrak{S}_\alpha^\beta\}$ will be called **τ-branching** if, for every vertex $e_\alpha \in |\alpha|$, there is a $\beta = \beta(\alpha) > \alpha$ such that, under the projection $\mathfrak{S}_\alpha^\beta : |\beta| \to |\alpha|$, exactly τ vertices $e_\beta \in |\beta|$ are projected onto the vertex e_α. In accordance with this definition, the definition of a **τ-branching sequence of partitions** $\Sigma = \{\alpha\}$ is introduced.

Theorem 2. *In order that the space X be homeomorphic to the Baire space B^τ, it is necessary and sufficient that one of the following conditions be fulfilled:*

a) *The space X is a metrizable complete space of dimension $\dim X = 0$ and of weight τ, every canonical closed subset of which has weight τ.*

b) *In the space X of weight τ there is a countable complete weakly refining sequence of partitions $\Sigma = \{\alpha_i\}$ of multiplicity 1, consisting of elements $A_t^i \in \alpha_i$ of the same weight (for every i).*

c) *In the space X there is a countable, complete, weakly refining τ-branching sequence of partitions $\Sigma = \{\alpha_i\}$ of multiplicity 1.*

d) *The space X is a space of a τ-branching zero-dimensional countable spectrum*

$$S = \{|\alpha_i|, \mathfrak{S}_i^{i+1}\}.$$

Remark. After this theorem it is easy to formulate necessary and sufficient conditions for X to be homeomorphic to an everywhere dense subset of the generalized Baire space.

The same method proves

Theorem 3. The following conditions imposed on the space are equivalent:

a) The space X is a perfect irreducible image of the entire Baire space B^τ.

b) The space X is a complete metric space of weight τ, containing no canonically closed subsets of weight $\tau' < \tau$.

c) In the space X of weight τ there is a countable complete weakly refining sequence of decompositions $\Sigma = \{\alpha_i\}$ containing no sets $A^i \in \alpha_i$ of smaller weight (for any i).

d) In the space X there is a countable complete τ-branching weakly refining sequence of decompositions $\Sigma = \{\alpha_i\}$.

e) The space X is a space $\widetilde{S}$ of a countable τ-branching spectrum

$$S = \{|\alpha_i|,\ \omega_i^{i+1}\}.$$

Theorem 4. The following conditions imposed on the space X are equivalent:

a) The space X is metrizable and $\dim X = n$.

b) In the metrizable space X there is a countable system of open locally finite coverings $\omega_i = \{u^i\}$ of multiplicity $n+1$, such that the covering $\bar{\omega}_{i+1}$ (consisting of the closures of the elements of the covering ω_i) is inscribed in the covering ω_i, and in some metric ρ of the space X the diameters of all $u^i \in \omega_i$ are less than some ε_i, where $\varepsilon_i \to 0$ (one may even put $\varepsilon_i = \frac{1}{i}$).

c) In the metrizable space X there is a countable set $\Sigma = \{\alpha_i\}$ of decompositions of multiplicity $n+1$ such that $\operatorname{diam} A^i < \frac{1}{i}$ for every $A^i \in \alpha_i$ in some metric ρ of the space X (then such a sequence of decompositions exists also for any other metric of the space X).

d) The space X is homeomorphic to a subset of the space $\widetilde{S}$ of a countable n-dimensional projection spectrum

$$Z = \{|\alpha_i|,\ \omega_i^{i+1}\}.$$

e) The space X is an $(n+1)$-fold image of some zero-dimensional (in the sense of dim) metric space under a perfect irreducible mapping.

Remark. The equivalence of a number of the conditions listed above was proved earlier by Dowker (4): a) ↔ b), and Morita (5): a) ↔ e).

Theorem 5. The following properties of a topological space are equivalent:

a) The space X admits a perfect mapping onto a metric space Y.

b) The space X is paracompact, and in it there exists such a sequence of open coverings ω_i that for every point $x_0 \in X$ there is a bicompact set A_{x_0} containing this point x_0, such that the sequence of coverings ω_i refines with respect to* this bicompact set A_{x_0}.

c) In the space X there is a countable sequence ω_i of open coverings such that the covering ω_{i+1} is star-inscribed in the covering ω_i for every i, and for each point $x_0 \in X$ there exists a bicompact set $A_{x_0} \ni x_0$ with respect to which this sequence refines.

d) In the space X there is such a countable sequence of decompositions that for every point $x_0 \in X$ there exists a bicompact set $A_{x_0} \ni x_0$ with respect to which the sequence refines.

Remark. The equivalence of a) and b) was proved simultaneously and independently also by A. Arhangel' skii (3).

The implications a) → b) → c) are proved without difficulty.

The implication c) → a) is proved as follows. Take a countable sequence of decompositions $\Sigma = \{\alpha_i\}$ and construct the metric space

$$Y = \widetilde{S}_0 = \underset{\leftarrow}{\lim}\{|\alpha_i|,\ \mathfrak{D}_i^{i+1}\}.$$

Let $\xi = \{t_\alpha\}$ be a maximal thread of the maximal spectrum $S = \{|\alpha|,\ \mathfrak{D}_\alpha^{\alpha'}\}$ of the space X (see (6,7)), determining the point x_0 of the space X; associate with it the thread $\eta = \{t_{\alpha_i}\}$ of the spectrum

$$S_0 = \{|\alpha_i|,\ \mathfrak{D}_i^{i+1}\},$$

contained in it. It turns out that the thread $\eta = \{t_{\alpha_i}\}$ is a maximal thread of the spectrum S_0, i.e., a point $y_0 = \eta$ of the space $\widetilde{S}_0 = Y$, and the mapping $f\xi = fx_0 = \eta = y_0$ into the metric space Y is a perfect mapping of X. This theorem is proved analogously.

Theorem 6*. *In order that the space X admit a perfect irreducible mapping f onto a metric space, it is necessary and sufficient that there exist a count-*

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* We say that a set $\mathfrak{A} = \{\omega\}$ of coverings refines with respect to a closed set $A \subset X$ if for every neighborhood UA of this set there is such an $\omega \in \mathfrak{A}$ that $\Gamma_\omega A \subset UA$, where $\Gamma_\omega A$ is the star of the set A with respect to the covering ω, i.e., the sum of all elements $U \in \omega$ for which $U \cap A \neq \Lambda$.

able sequence of decompositions $\{\alpha_i\}$ *satisfying simultaneously the following two conditions:*

a) *Condition d), formulated in Theorem 5.*

b) *For every open set* U *in* X *there is, for some* i, *an* $x_0 \in U$ *such that* $O_{\alpha_i} x_0 \subset U$.

Corollary. Theorem 6 gives a necessary and sufficient condition for there necessarily to be a metric space in the class of spaces with homeomorphic absolutes (see ([6],[7])).

Remark 1. If in the space X one requires the existence of a sequence $\Sigma = \{\alpha_i\}$ of decompositions of multiplicity $n+1$ satisfying condition d), formulated in Theorem 5, then one obtains a necessary and sufficient condition for the space X to admit a perfect mapping onto a metric space Y of dimension $\dim Y = n$.

The proof is analogous to the proof of Theorem 5.

Remark 2. There exist paracompacta, and even bicompacta, that do not admit any irreducible perfect mapping onto a metric space. Such bicompacta are, for example, dyadic nonmetrizable bicompacta, as well as the Čech growth of an uncountable number of isolated points. Moreover, these bicompacta have no everywhere dense sets admitting an irreducible perfect mapping onto a metric space.

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* It is not difficult to formulate Theorems 5 and 6 in spectral terms.

Note: Figure translations are in progress. See original paper for figures.