Mathematics

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Abstract

Full Text

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ON LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. We first note one simple comparison theorem. As is known, in the case of nonoscillation of the equation $\ddot{x} + q(t)x = 0$ (i.e., in the case when its nontrivial solutions have a finite number of zeros on $(0, \infty)$), there exists a solution $x(t)$, unique up to a factor, such that

$$\int_{t_0}^{\infty} x^{-2}(t)\, dt = \infty$$

(t_0 is chosen to the right of the zeros of $x(t)$); this solution is naturally called **minimal**. We shall call a fundamental system $x_1(t), x_2(t)$ a **proper fundamental system** (abbreviated p.f.s.) if $x_2(t)$ is a minimal solution and $x_1(t), x_2(t) > 0$ for $t \geq t_0$. Consider the equations

$$\ddot{x} + q(t)x = 0, \tag{1}$$

$$\ddot{y} + r(t)y = 0. \tag{2}$$

Theorem 1. Let $q(t) \leq r(t)$ $(t \geq t_1)$, and let equation (2) be nonoscillatory. Then the p.f.s.' s $x_1(t), x_2(t)$ and $y_1(t), y_2(t)$ for (1) and (2), respectively, satisfy the conditions:

$$x_1(t) \geq C_1 y_1(t), \qquad x_2(t) \leq C_2 y_2(t) \qquad (C_1, C_2 > 0,\ t \geq t_0).$$

Examples: If $q(t) \leq -a^2$ $(a > 0,\ t \geq t_1)$, then any p.f.s. is such that $x_1(t) \geq C_1 e^{at}$, $x_2(t) \leq C_2 e^{-at}$ $(t \geq t_0)$; if $q'(t) \leq k(1-k)t^{-2}$ $(k > 1/2,\ t \geq t_1)$, then any p.f.s. is such that $x_1(t) \geq C_1 t^k$, $x_2(t) \leq C_2 t^{1-k}$ $(t \geq t_0)$, and so on. With the help of Theorem 1 it is also easy to show that if $q(t)$ is positive and nondecreasing for $t \geq t_1$, then any p.f.s. for the equation $\ddot{x} - q(t)x = 0$ is such that

$$x_1(t) \geq C_1 \exp\left(\int_{t_0}^{t} \sqrt{q(s)}\, ds\right), \qquad x_2(t) \geq C_2 \exp\left(-\int_{t_0}^{t} \sqrt{q(s)}\, ds\right) \quad (t \geq t_0). \tag{3}$$

2. Consider the antiperiodic boundary-value problem

$$\ddot{x} + [q(t) + \lambda]x = 0, \qquad x(a) = -x(b), \qquad \dot{x}(a) = -\dot{x}(b). \tag{4}$$

Theorem 2. The smallest eigenvalue λ_1 of the boundary-value problem (4) satisfies the inequality*

$$\lambda_1 \leq \frac{\pi^2}{(b-a)^2} - \frac{I}{b-a}, \qquad \text{where } I = \int_a^b q(t)\, dt. \tag{5}$$

* As M. G. Krein informed the author, he has also obtained some analogous estimates.

Estimate (5) is attained for $q(t) \equiv$ const; it shows, among other things, that in the case $q(t) \geqslant 0$, using the single characteristic I, one can give a two-sided estimate for the first positive eigenvalue μ_1 of the boundary-value problem $\ddot{x} + \mu q(t)x = 0,\ x(a) = -x(b),\ \dot{x}(a) = -\dot{x}(b)$:

$$4 < \mu_1(b-a)I \leqslant \pi^2$$

(note that, for example, for the periodic problem or the problem $x(a) = x(b) = 0$, such an estimate is impossible).

3. For many questions (estimates of eigenvalues, applicability of Chaplygin' s theorem, etc.) conditions are of interest under which every nontrivial solution of the equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{6}$$

has no more than one zero on the given interval $[a, b]$. In this connection various characteristics of the coefficients may be used; the criterion given below uses the norms of the coefficients in $L_1[a, b]$, and Sturm' s theorem makes it possible, instead of the norm of $q(t)$, to restrict oneself to the norm of the function $q_+(t) = \max\{0, q(t)\}$.

Theorem 3. *If the inequality*

$$I_1 e^{I_2/2} \leqslant \frac{4}{b-a}, \qquad \text{where } I_1 = \int_a^b q_+(t)\, dt, \qquad I_2 = \int_a^b |p(t)|\, dt,$$

is satisfied, then every nontrivial solution of equation (6) has no more than one zero on $[a, b]$.

The formulated criterion is unimprovable in the characteristics I_1, I_2.

4. Put

$$I(p) = \int^{\infty} \exp\left(-\int^{t} p(s)\,ds\right) dt, \qquad K(p,q) = \int^{\infty} q(t) \exp\left(\int^{t} p(s)\,ds\right) dt$$

(the lower limits may be taken to be arbitrary finite ones, since we shall be interested only in questions of convergence). Sturm' s theorem shows that a decrease of the "attraction" $q(t)$ preserves nonoscillation, i.e., from the nonoscillation of (6) follows the nonoscillation of the equation

$$\ddot{x} + p_1(t)\dot{x} + q_1(t)x = 0, \tag{7}$$

if $p_1(t) \equiv p(t)$, $q_1(t) \leqslant q(t)$. The natural question is how, in this respect, a change of the "friction" $p(t)$ affects the matter.

For the case $q(t) \geqslant 0$ the answer is given by the following propositions.

Theorem 4. *Let* $q(t) \geqslant 0$. *If* $I(p) < \infty$ $(= \infty)$, *then an increase (decrease) of* $p(t)$ *preserves nonoscillation. More precisely, if* $q_1(t) \leqslant q(t)$, *then the nonoscillation of (6) implies the nonoscillation of (7) in either of the following cases: a)* $I(p) < \infty$, $p_1(t) \geqslant p(t)$; *b)* $I(p) = \infty$, $p_1(t) \leqslant p(t)$.

Theorem 5. *Let* $q(t) \geqslant 0$, $I(p) < \infty$, $K(p,q) < \infty$. *Then both an increase and a decrease of* $p(t)$ *preserve nonoscillation, i.e., if* $q_1(t) \leqslant q(t)$ *and either of the inequalities* $p_1(t) \geqslant p(t)$, $p_1(t) \leqslant p(t)$ *is satisfied, then the nonoscillation of (6) implies the nonoscillation of (7).*

We note that, in particular, $I(p) = \infty$ if $p(t) \leqslant t^{-1}$, and $I(p) < \infty$ if $p(t) \geqslant (1+\varepsilon)t^{-1}$ $(\varepsilon > 0)$ for sufficiently large t.

5. Consider the equation with a parameter

$$\ddot{x} + \alpha p(t)\dot{x} + q(t)x = 0, \qquad (p(t), q(t) \geqslant 0,\ -\infty < \alpha < \infty). \tag{8}$$

Denote by α_0 $(\leqslant \infty)$ the infimum of those degrees for which the function

$$h(t) = \exp\left(-\int^{t} p(s)\,ds\right)$$

is summable on $(0, \infty)$. Theorem 4 leads, in particular, to the following "extrapolation" assertion: in order that equation (6) be nonoscillatory for every α, it is necessary and sufficient that it be nonoscillatory in some neighborhood of the point $\alpha = \alpha_0$. It is curious that the point of "greatest oscillation" α_0 is determined by the single function $p(t)$, and moreover in a quite effective way.

Theorem 6. In the case of nonoscillation of (6), the condition $I(p) < \infty$ is necessary, and, when $q(t) \geqslant 0$, sufficient for the boundedness of all solutions of (6) on $(0, \infty)$.

This proposition is a generalization of a close result from ([2]). Comparing Theorems 4 and 6 leads to the following comparison theorem.

Theorem 7. Let $0 \leqslant q_1(t) \leqslant q(t)$, $p_1(t) \geqslant p(t)$. If the solutions of (6) do not oscillate and are bounded for $t \geqslant 0$, then the solutions of (7) also do not oscillate and are bounded for $t \geqslant 0$.

6. Let the coefficients of the equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = f(t) \tag{9}$$

satisfy the conditions

$$0 \leqslant m \leqslant q(t) \leqslant M, \qquad p(t) \geqslant l > 0 \qquad (t \geqslant t_0). \tag{10}$$

As the following propositions show, some of the results formulated in ([2]) for equation (6) extend to the case of the nonhomogeneous equation (9).

Below we use the function $\psi(l, m, M)$ defined in ([2]) and the transcendental constant $h_0 \approx 3.046$.

Theorem 8. Let conditions (10) be fulfilled, and suppose that

$$\psi(l, m, M) < 1. \tag{11}$$

Then for any $f(t) = O(1)$ all solutions of (9), together with their derivatives, are bounded on $(0, \infty)$.

Theorem 9. Let conditions (10), (11) be fulfilled and, in addition,

$$\int^{\infty} dt \int^{t} \exp\left(-\int_s^t p(\tau)\, d\tau\right) ds = \infty. \tag{12}$$

Then for any $f(t) = O(1)$ all solutions of (9), together with their derivatives, tend to zero as $t \to \infty$.

Of course, first derivatives are meant. Condition (12), first used by Opial in ([1]), is a restriction, in a certain sense, on the growth of $p(t)$ as $t \to \infty$; it is not difficult to show that, in particular, it is fulfilled when $p(t) = O(t)$.

7. We now consider, instead of (10), the less restrictive conditions

$$0 \leqslant q(t) \leqslant M, \qquad p(t) \geqslant l > 0 \qquad (t \geqslant t_0). \tag{13}$$

Theorem 10. Let conditions (13) be fulfilled, and suppose that

$$M < h_0 l^2. \tag{14}$$

Then for any $f(t) = O[q, t]$ all solutions of (9), together with their derivatives, are bounded on $(0, \infty)$.

Theorem 11. Suppose that conditions (13), (14) are satisfied and, moreover,

$$\int^{\infty} dt \int^{t} q(s) \exp\left(-\int_{s}^{t} (-p(\tau)\, d\tau)\right) ds = \infty. \tag{15}$$

Then for any $f(t) = o[q(t)]$ all solutions of (9), together with their derivatives, tend to zero as $t \to \infty$.

The notation $f(t) = O[q(t)]$ and $f(t) = o[q(t)]$ means, respectively, that $|f(t)| \leq Cq(t)$ $(t \geq t_1)$ and $f(t) = \varepsilon(t) q(t)$, $\varepsilon(t) \to 0$ as $t \to \infty$. We note that condition (15) (considered earlier in ([2],[3])) is satisfied, in particular, if $q(t) \geq C_1 t^{-r}$, $p(t) \geq C_2 t^{1-r}$ $(C_1, C_2 > 0,\ 0 \leq r \leq 1,\ t \geq t_0)$.

8. Theorems 8-11 cannot be improved with respect to the characteristics l, m, M. Condition (11) in Theorems 8 and 9 may be replaced by the simpler-to-check (but more restrictive) conditions $\sqrt{M} - \sqrt{m} \leq \pi l/2$ or $M \leq h_0 l^2$. Analogous results can also be obtained for certain nonlinear equations; in particular, Theorems 8-11 remain valid for the equation

$$\ddot{x} + p(t, x, \dot{x})\dot{x} + q(t, x, \dot{x})x = f(t, x, \dot{x}),$$

if the functions p, q, f satisfy the corresponding requirements uniformly in $x, \dot{x}$.

The proof of Theorems 8-11 is based on reducing the questions under consideration to a certain optimal-control problem; in solving the latter, the mechanical interpretation of (9) as the equation of one-dimensional oscillations with variable friction coefficient $p(t)$, variable attraction coefficient $q(t)$, and disturbing force $f(t)$ is heuristically useful.

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Note: Figure translations are in progress. See original paper for figures.