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Abstract

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MATHEMATICAL PHYSICS

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ON THE SEPARATION OF SELF-ACTION AND SCATTERING EFFECTS IN PERTURBATION THEORY

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As is well known, the physical basis of the renormalization method in quantum field theory is the desire to deal only with observable quantities. In this sense the usual renormalization scheme has, as was noted by Van Hove ([1]), the following shortcoming: it makes extensive use of the expansion in eigenfunctions of the free Hamiltonian, which entails the necessity of introducing wave-function renormalization factors that have no clear intuitive meaning.

On the basis of studying the perturbation-theory series for the resolvent of the full Hamiltonian, Van Hove constructed states that determine the asymptotics, for large $|t|$, of solutions of the nonstationary Schrödinger equation. Self-action effects are exactly taken into account in these states. In the present work another approach is proposed to the problem of excluding the eigenfunctions of the free Hamiltonian from calculations. It is shown that, by means of a unitary transformation, one can reduce the energy operator to the sum of two terms, one of which includes all self-action effects, while the other contains only the terms responsible for scattering. The eigenfunctions of the first term may be taken as the asymptotic functions. The arguments are based on perturbation theory and are not mathematically rigorous.

1. The proposed approach will be illustrated by the example of a system of interacting neutral scalar mesons. The creation and annihilation operators $a^*(k)$ and $a(k)$ satisfy the usual commutation relations

$$[a(k), a(k')] = [a^*(k), a^*(k')] = 0; \qquad [a(k), a^*(k')] = \delta(k - k'). \tag{1}$$

We shall assume that the energy operator has the form

$$H = H_0 + \varepsilon V; \qquad V = \sum_{\alpha,\beta} V_{\alpha\beta}, \tag{2}$$

where

$$H_0 = \int \omega(k) a^*(k) a(k)\, dk, \tag{3}$$

$$V_{\alpha\beta} = \int v_{\alpha\beta}(k_1, \dots, k_\alpha;\, k'_1, \dots, k'_\beta)\delta(k_1 + \dots + k_\alpha - k'_1 - \dots - k'_\beta) \times$$

$$\times a^*(k_1) \cdots a^*(k_\alpha) a(k'_1) \cdots a(k'_\beta)\, dk_1 \cdots dk_\alpha\, dk'_1 \cdots dk'_\beta, \tag{4}$$

and the summation in (2) is over values $\alpha + \beta \geq 3$. The coefficient functions $v_{\alpha\beta}$ satisfy the symmetry condition

$$v_{\alpha\beta}(k_1, \dots, k_\alpha;\, k'_1, \dots, k'_\beta) = \overline{v_{\beta\alpha}(k'_1, \dots, k'_\beta;\, k_1, \dots, k_\alpha)}. \tag{5}$$

In what follows, operators of the form (4) will be called operators of type (α, β), according to the number of operators $a^*(k)$ and $a(k)$ in the corresponding product.

For the applicability of nonstationary scattering theory it is necessary that the asymptotic condition be satisfied in the following form: the operator

$$U(t) = \exp\{iHt\} \exp\{-iH_0 t\}$$

must have strong limits as $t \to \pm\infty$. We note here that often, using the adiabatic hypothesis, one considers limits of the type

$$U^{(\pm)} = \lim_{\varepsilon \to \pm 0} \mp\varepsilon \int_0^{\mp\infty} e^{\varepsilon t} U(t)\, dt.$$

Such limits exist for a broad class of operators H_0 and H; however, the limiting operators $U^{(\pm)}$, and with them the corresponding scattering operator $S = U^{(+)*}U^{(-)}$, will not, generally speaking, be unitary. It is precisely with this that the appearance of wave-function renormalization factors in the usual perturbation theory is connected.

We shall show that, for the asymptotic condition to be fulfilled, it is necessary that the summation in (2) begin with the values $\alpha \geqslant 2$ and $\beta \geqslant 2$, i.e., that the interaction, expanded in a series of normal products, contain at least two creation operators and two annihilation operators.

For this purpose let us note that if the interaction contains terms $(\alpha, 0)$ or $(\alpha, 1)$, then the operator $U(t)$ cannot converge on the subspace containing the free one-particle states and the vacuum. Indeed, if $U(t)\Phi$ converges, then the norm of the element $\Phi' = V \exp\{-iH_0 t\}\Phi$ must vanish as $|t| \to \infty$. Let Φ_0 be the vacuum, and Φ_f a one-particle state, i.e.

$$\Phi_f = \int f(k) a^*(k)\, dk\, \Phi_0$$

and

$$\Phi_\alpha(t) = V_{\alpha 1} \exp\{-iH_0 t\}\Phi_f.$$

It is not difficult to calculate that

$$\Phi_\alpha(t) = \int f_\alpha(k_1, \dots, k_\alpha; t) a^*(k_1) \cdots a^*(k_\alpha)\, dk_1 \cdots dk_\alpha\, \Phi_0,$$

where

$$f_\alpha(k_1, \dots, k_\alpha; t) = v_{\alpha 1}(k_1, \dots, k_\alpha; k') f(k') \exp\{-i\omega(k')t\}; \quad k' = k_1 + \dots + k_\alpha.$$

The norm of the element $\Phi_\alpha(t)$ in general does not depend on t. Thus, if the asymptotic condition is fulfilled, then terms of type $(\alpha, 1)$ in the interaction must be absent. In an analogous way one can see that the interaction must not contain terms of type $(\alpha, 0)$. From the symmetry condition (5) it follows that terms of type $(1, \beta)$ and $(0, \beta)$ must also be absent. The necessary condition formulated follows from the preceding arguments.

2. The Hamiltonians with which one has to deal in field theory do not satisfy this condition. We shall show, however, that if the function $\omega(k)$ satisfies the convexity condition

$$\omega(k_1) + \omega(k_2) > \omega(k_1 + k_2), \tag{6}$$

then one can choose such a unitary operator W that

$$H' = W^{-1} H W = c + H_0' + V',$$

where c is a constant, H_0' has the form (3) with a function $\omega'(k)$ that is, generally speaking, different from $\omega(k)$, and V' is represented by a series of normal products in which the number of creation and annihilation operators is not less than two.

The arguments are based on perturbation theory. We shall seek the operators W and H' in the form of expansions in the parameter ε:

$$W = I + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots; \quad H' = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots.$$

We substitute these expansions into the relations $HW = WH'$, $W^*W = I$, and equate the coefficients of equal powers of ε.

in powers of ε. We obtain a system of relations

$$W_1^* + W_1 = 0, \qquad W_n + W_n^* + \sum_{k=1}^{n-1} W_k^* W_{n-k} = 0, \qquad n \geqslant 2; \tag{7}$$

$$[H_0, W_1] + V = H_1; \tag{8^1}$$

$$[H_0, W_n] + VW_{n-1} = \sum_{k=1}^{n-1} W_k H_{n-k} + H_n, \qquad n \geqslant 2. \tag{8^2}$$

Let B_n and A_n be the symmetric and antisymmetric parts of the operator W_n. Relations (7) will be satisfied if arbitrary operators are taken as A_n, while B_n is determined from the recurrence relations

$$B_1 = 0; \qquad 2B_n = \sum_{k=1}^{n-1} (A_k - B_k)(A_{n-k} + B_{n-k}), \qquad n \geqslant 2.$$

If the W_n constructed in this way are substituted into (8), these relations take the form

$$[H_0, A_n] + Q_n = H_n, \qquad n = 1, 2, \dots, \tag{9}$$

where Q_n is a symmetric operator which is explicitly expressed in terms of the operators V, A_k, and H_k for $k < n$. We use (9) for the recurrent determination of H_n and A_n. Suppose that we know A_k and H_k for $k = 1, \dots, n-1$. The operator Q_n is then also known. Denote by $\widetilde{Q}_n$ the sum of the terms of the types $(\alpha, 0)$, $(0, \beta)$, $\alpha, \beta \geqslant 1$, and $(\alpha, 1)$, $(1, \beta)$, $\alpha, \beta \geqslant 2$, in the series of normal products for Q_n, and put $H_n = Q_n - \widetilde{Q}_n$. The operator H_n, generally speaking, contains a constant, a term of type $(1, 1)$, and terms of type (α, β) with $\alpha \geqslant 2$ and $\beta \geqslant 2$, giving an n-th order contribution to the constant c and to the operators H_0' and V', respectively.

To determine A_n, we are left with the relation

$$[H_0, A_n] + \widetilde{Q}_n = 0. \tag{10}$$

As its solution we propose to take the operator A_n, represented by the sum of terms of the same type as the operator $\widetilde{Q}_n$. Denote by $a_{\alpha\beta}$ and $q_{\alpha\beta}$ the corresponding coefficient functions for the operators A_n and $\widetilde{Q}_n$. From (10) it follows that

$$a_{\alpha\beta}(k_1, \dots, k_\alpha;\, k'_1, \dots, k'_\beta) = [\omega(k_1) + \dots + \omega(k_\alpha) - \omega(k'_1) - \dots - \omega(k'_\beta)]^{-1} \times q_{\alpha\beta}(k_1, \dots, k_\alpha;\, k'_1, \dots, k'_\beta). \tag{11}$$

In the integrals defining A_n, the functions $a_{\alpha\beta}$ enter only when

$$k_1 + \dots + k_\alpha = k'_0 + \dots + k'_\beta.$$

By the definition of the operator $\widetilde{Q}_n$, the numbers α and β take unequal values, and only one of them can differ from 0 or 1. For such $\alpha, \beta, k_1, \dots, k_\alpha, k'_1, \dots, k'_\beta$, the denominator in (11) does not vanish by virtue of condition (6), so that relation (11) determines $a_{\alpha\beta}$ uniquely. From the symmetry condition of type (5) for the functions $q_{\alpha\beta}$ it follows that

$$a_{\alpha\beta}(k_1, \dots, k_\alpha;\, k'_1, \dots, k'_\beta) = -\overline{a_{\beta\alpha}(k'_1, \dots, k'_\beta;\, k_1, \dots, k_\alpha)},$$

i.e., the operator A_n is antisymmetric. Thus the proposed scheme is self-consistent.

3. For the operators H'_0 and $H' - c$ there are no obstacles, discussed in Sec. 1, to the fulfillment of the asymptotic condition. It should be expected that

under very broad conditions on the coefficient functions $v_{\alpha\beta}$, there exist strong limits of the operator

$$U'(t) = \exp\{i(H' - c)t\} \exp\{-iH'_0 t\}$$

as $t \to \pm\infty$. It is natural to take as the scattering operator the operator

$$S = {U'}^*(+\infty) U'(-\infty).$$

Its matrix elements between the eigenfunctions of the operator H'_0 have physical meaning.

From the considerations of Sec. 2 there follows a scheme of perturbation theory for constructing the scattering operator that differs from that accepted in field theory. Namely, the calculations are divided into two stages: at the first one should explicitly separate in the Hamiltonian the contributions responsible for the effects of self-action and scattering, and at the second one should calculate the scattering operator according to the usual scheme, starting from an unperturbed operator that includes all self-action effects. With this method the S-matrix is obtained at once as unitary and has nontrivial matrix elements only between states containing no fewer than two particles. In particular, the need to renormalize wave functions disappears.

In theories with local interactions, divergences appear at the first stage and are removed by mass renormalization, and at the second by charge renormalization.

In the concrete construction of the operator H', it is convenient to seek the operator W in a form that takes its unitarity into account in advance, for example, $W = \exp\{R\}$; $W = (I + C)(I - C)^{-1}$, where R or C are antisymmetric operators. To determine these operators by perturbation theory one can construct recurrence relations of type (9). The expression for the operator H' does not depend on the specific choice of the form of the operator W.

4. Instead of speaking of a unitary transformation of the energy operator, one may, in equivalent terms, speak of a choice of representation for the operators $a^*(k)$ and $a(k)$ in which they are not creation and annihilation operators. Namely, if we consider the operators $a'(k) = Wa(k)W^{-1}$, ${a'}^*(k) = Wa^*(k)W^{-1}$, then the operator H, expressed through these operators, takes the form $H(a^*, a) = H'({a'}^*, a')$. It is natural to regard the operators ${a'}^*(k)$ and $a'(k)$ as the creation and annihilation operators.

In many interesting cases, in particular if the interaction contains terms of the type $(\alpha, 0)$ and $(0, \beta)$, the transformation W is, as is customary to say, an infinite unitary transformation. More precisely, in these cases the operators a, a^* and $a', {a'}^*$ realize inequivalent representations of the commutation relations (1).

In conclusion we note that the transformation W is a dressing transformation in the sense of Greenberg and Schweber [2]. The requirements given in [2] do not determine the dressing transformation uniquely. It seems to us that the transformation constructed in this work is minimal in the sense that it exactly takes into account only self-action effects.

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