MATHEMATICAL PHYSICS

1963

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

Yu. P. PYT' EV

ON THE CONNECTION BETWEEN CLASSICAL MECHANICS AND WAVE MECHANICS

(Presented by Academician I. G. Petrovskii on 18 X 1962)

As is known, the behavior of classical particles is determined by the Hamilton–Jacobi equation. Let us imagine that, when a particle moves, generally speaking, there occurs a process of propagation of some field, and that the observed classical particles correspond to singularities of this field. We must therefore assume that the behavior of the field singularities is determined by the Hamilton–Jacobi equation, and then the problem arises of reconstructing, from the Hamilton–Jacobi equation, the corresponding field equation (or system of equations).

It is known that the geometric locus of possible discontinuities of the highest derivatives (and, under certain natural assumptions, also of lower derivatives and of the solution itself) of a wave equation (or of a system of hyperbolic type) is given by the characteristic manifolds of this equation (see, for example, (1)). If the principal part of the wave equation is linear with respect to the highest derivatives, then it can be determined from the equation of the characteristic manifolds. The other part, generally speaking nonlinear with respect to the remaining derivatives and to the unknown function, remains arbitrary.

In the present work it is shown that if the Hamilton–Jacobi equation is taken as the equation of the characteristic manifold, then the simplest consequence of the propagation equation obtained is the Schrödinger equation in the nonrelativistic case and the Klein–Gordon equation in the relativistic one. Further, it is established by direct verification that the characteristic manifolds of the correspondingly transformed Dirac equation, as well as of the equations of the unified theory of electromagnetic and gravitational fields, are also described by the Hamilton–Jacobi equation. Thus the displacement of the front of discontinuities of the field is associated with the motion of a classical particle. The normal and ray velocities of displacement of the front in x-space coincide with the velocity of the particle. The field singularities located on the front propagate along the bicharacteristics of the propagation equation (classical trajectories); in this case the energy and momentum of the singular part of the field coincide with the energy and momentum of the particle.

1. The characteristic manifolds of partial differential equations of hyperbolic type (or hyperbolic systems) are described by first-order partial differential

equations, which can be represented in the form of the vanishing of a certain homogeneous form with respect to the partial derivatives. The order of the form is determined correspondingly by the order of the original equation (1).

The Hamilton–Jacobi equation for a particle in electromagnetic A_i and gravitational g^{ik} fields is

$$g^{ik}(s_i + g_i mc)(s_k + g_k mc) = -m^2c^2;$$

$$g_i = \frac{e}{mc^2}A_i; \qquad s_i = \frac{\partial s}{\partial x^i} \qquad i,k = 1,2,3,4, \tag{1}$$

by a well-known method can be transformed to the form of the eikonal equation in the space $x^1, x^2, x^3, x^4 = ict, x^5 = s/(mc)$ *:

$$G^{\mu\nu}\varphi_\mu\varphi_\nu = 0; \qquad G^{\mu\nu} = \begin{pmatrix} g^{ik} & -g^i \\ -g^k & 1 + g^{ik}g_ig_k \end{pmatrix}; \quad \mu,\nu = 1,2,\dots,5. \tag{2}$$

For a known integral $\varphi(x^\nu)$ of equation (2), the solution $s(x^i)$ of equation (1) is determined from the equality $\varphi(x^\nu) = 0$.

In accordance with the program described above, we assume that equation (2), or in a more general form the equation

$$\left(G^{\mu\nu}\varphi_\mu\varphi_\nu\right)^q = 0, \qquad q = 1,2,\dots, \tag{3}$$

describes the behavior of the singularities of a certain field. Let us consider what form the corresponding equations of propagation of the field have. In the simplest case of the motion of a nonrelativistic particle in a potential field $U(x,t)$, equation (2) has the form

$$-\varphi_s\varphi_t + \frac{1}{2m}(\nabla\varphi)^2 + U\varphi_s^2 = 0. \tag{4}$$

A propagation equation having (4) as the equation of the characteristic manifold can be written in the following way:

$$-W_{st} + \frac{1}{2m}\Delta W + UW_{ss} + R = 0. \tag{5}$$

Here R is an arbitrary function of x, t, s, W and of the first derivatives of W. It is not difficult to show that in the simplest case equation (5) implies the Schrödinger equation. Indeed, let us require that the function R not depend on s, W_s and be homogeneous of first degree with respect to W and its derivatives.

Then (5) admits solutions of the form $W(x,t,s) = \Psi(x,t)\exp\{-is/\hbar\}$, where Ψ satisfies an equation not containing s:

$$i\hbar\Psi_t + \frac{\hbar^2}{2m}\Delta\Psi - U\Psi + \hbar^2 R = 0. \tag{6}$$

Identifying $\hbar$ with Planck' s constant, it is not difficult to see that (6), up to terms of order $\hbar^2$, coincides with the Schrödinger equation. If in equation (2) one sets $g^{ik} = \delta^{ik}$ (the gravitational field is absent), then the corresponding equation of the W-field can be written as follows:

$$\Delta W - 2\frac{e}{c}(\mathbf{A}\nabla)W_s + \left(\frac{e^2}{c^2}\mathbf{A}^2 - e^2\Phi^2 + m^2c^2\right)W_{ss} - \frac{2e}{c}\Phi W_{st} - \frac{1}{c^2}W_{tt} + R = 0,$$

and the same substitution, with analogous requirements on R, leads, as is easily verified, to the Klein–Gordon equation (up to terms of order $\hbar^2$).

Let us now consider the equations of characteristic manifolds for the Dirac equations and for a unified theory of electromagnetic and gravitational fields. Assuming that $-i/\hbar$ in the Dirac equation

$$\left(-\frac{i}{\hbar}\right)^2 \alpha_0 mc\Psi + \vec{\alpha}\left\{-\frac{i}{\hbar}\nabla\Psi - \frac{e}{c}\mathbf{A}\left(-\frac{i}{\hbar}\right)^2\Psi\right\} + e\Phi\left(-\frac{i}{\hbar}\right)^2\Psi = \frac{i}{\hbar c}\Psi_t$$

* In what follows we adhere to the generally accepted summation convention; Greek indices range from 1 to 5.

appears as a consequence of differentiation with respect to s; we obtain the equation for $W = \frac{\partial}{\partial s}\{\Psi\exp(-is/\hbar)\}$

$$\alpha_0 mcW_s + \vec{\alpha}\left\{\nabla W - \frac{e}{c}\mathbf{A}W_s\right\} + e\Phi W_s + \frac{1}{c}W_t = 0.$$

The equation of the characteristic manifolds of this system can be represented in the form of the equality to zero of the determinant

$$\det\left\{\varphi_s\left(\alpha_0 mc - \vec{\alpha}\frac{e}{c}\mathbf{A} + e\Phi\right) + \vec{\alpha}\nabla\varphi + \frac{1}{c}\varphi_t\right\} = 0,$$

which, as is not difficult to verify, gives equation (3) for $q = 2$. As for the equations of the unified five-dimensional theory of the electromagnetic and gravitational fields ([2])

$$R_{\mu\nu} - \frac{1}{2}\hat{G}_{\mu\nu}R = +\chi Q_{\mu\nu}; \qquad \hat{G}_{\mu\nu} = \begin{pmatrix} (1+\chi)(q_{ik}+\hat{g}_i\hat{g}_k) & (1+\chi)\hat{g}_k \\ (1+\chi)\hat{g}_i & 1+\chi \end{pmatrix};$$

$$\hat{g}_i = \sqrt{\frac{\chi}{2\pi}}\, A_i,$$

then in form they coincide with the equations of Einstein' s theory of gravitation, for which the form of the equation of characteristic manifolds is well known ([3]). Using this, one can write the sought equation in the form $G^{\mu\nu}\varphi_\mu\varphi_\nu = 0$. According to ([2]), this equation should be considered not in universal space, but in configuration space, i.e. everywhere in $\hat{G}^{\mu\nu}$ one must replace $\sqrt{\chi/2\pi}$ by e/mc^2, after which it is not difficult to see that the equation obtained coincides with (2).

2. Let the singularity of the W-field be located on the hypersurface $\varphi = 0$ in the space of r variables x^ν. In the space of p variables $x^1x^2x^3x^5$ this characteristic manifold is the aggregate of all surfaces $t = t(x^1, x^2, x^3, x^5)$, which may also be represented as a surface moving as t changes. Thus we are dealing with a solution $W(x^\nu)$ of the propagation equation, to which there corresponds a discontinuity front moving in the space p. For the normal velocity of motion of the front it is not difficult to obtain the expression $\mathbf{u}_p = \nabla_p t \cdot (\nabla_p t)^2$. In the x-space of the variables x^1, x^2, x^3 the velocity of motion of the front is given by the expression: $\mathbf{u}_x = \nabla_x t \cdot (\nabla_p t)^2$, which, as is not difficult to verify, coincides with the velocity of the particle:

$$\mathbf{v} = c^2\frac{\mathbf{P}}{E} = -\nabla_x s \cdot s_t \left((mc)^2 + (\nabla_x s)^2\right)^{-1} = \mathbf{u}_x$$

(by virtue of $\varphi = 0$). It is not difficult, moreover, to verify that with the same velocity the singularity moves along the bicharacteristics of the propagation equation of the W-field (i.e. along the classical trajectories–the characteristics of equation (2)), since in x-space the vectors $\nabla_p t$ and $\dot{x}^\mu = 2G^{\mu\nu}\varphi_\nu$ are collinear.

Let us consider the question of the magnitude of the discontinuity of the W-field. Let $x^\nu = x^\nu(\alpha^\mu, \lambda)$ be a family of characteristics of equation (2) in r-space (λ is the parameter along the characteristics, α^μ are integrals of the system $\dot{x}^\mu = 2G^{\mu\nu}\varphi_\nu$). It is not difficult to verify that for the determinant $\Theta = |\partial x^\mu/\partial\alpha^\nu|$ the relation holds

$$\dot{\Theta} = \Theta\frac{\partial}{\partial x^\nu}\left(2G^{\mu\nu}\varphi_\mu\right) = 2\Theta\left\{\frac{\partial G^{\mu\nu}}{\partial x^\nu}\varphi_\mu + G^{\mu\nu}{\varphi_\mu}^\nu\right\}. \tag{7}$$

Assuming for simplicity $g^{ik} = \delta^{ik}$, we obtain $\partial G^{\mu\nu}/\partial x^\nu = 0$ (Lorentz gauge for the electromagnetic potentials). On the other hand, assuming that W satisfies a second-order equation (the case of a system is treated analogously) and discarding the nonlinear supplement, we obtain an equation for the magnitude of the discontinuity π of the W-field: $\ddot{\pi} + \pi \cdot G^{\mu\nu}\varphi_{\mu\nu} = 0$ ([1]). Hence, from (7), it follows that

$$\pi = \rho\Theta^{-1/2}, \qquad \rho = \rho(\alpha^\mu). \tag{8}$$

3. Let us consider, using equation (5) as an example, the question of the energy and momentum of the singular part of the W-field. It is not difficult to verify that equation (5) corresponds to a Lagrangian of the following form:

$$\mathcal{L} = -W_s W_t + \frac{1}{2m}(\nabla W)^2 + UW_s^2. \tag{9}$$

Writing out, with the help of (9), the values of the components of the energy-momentum tensor: $T_{44} = \dfrac{(\nabla W)^2}{2m} + UW_s^2$, $T_{i4} = W_{x^i} W_s$, we obtain the following expressions for the energy and momentum of the W-field:

$$\mathcal{E} = \iiiint \left\{ \frac{(\nabla W)^2}{2m} + UW_s^2 \right\} d\Omega_p; \qquad \mathcal{P}_i = -\iiiint W_{x^i} W_s \, d\Omega_p. \tag{10}$$

Let us refer the region of integration in (10) to new coordinates, as which we take the integrals $\alpha^1, \alpha^2, \alpha^3, \alpha^5 = \varphi$ (for example, $\alpha^i = x^i_{\text{init}}$) of the characteristic system of equation (4). In the new coordinates

$$W_{x^\nu} = W_\varphi \varphi_\nu + \cdots,$$

$$d\Omega_p = dx^1\, dx^2\, dx^3\, dx^5 = \left| \frac{\partial(x^i, x^5)}{\partial(\alpha^i, \varphi)} \right| d\alpha^1\, d\alpha^2\, d\alpha^3\, d\varphi = \hat{\theta}\, d\hat{\Omega},$$

and for the integrals (10) we obtain

$$\mathcal{E} = \iiint\!\!\int_{\hat{\Omega}} \left\{ \left[\frac{(\nabla\varphi)^2}{2m} + U\varphi_s^2 \right] W_\varphi^2 + \cdots \right\} \hat{\theta}\, d\hat{\Omega};$$

$$\mathcal{P}_i = -\iiint\!\!\int_{\hat{\Omega}} \left\{ \varphi_{x^i} \varphi_s W_\varphi^2 + \cdots \right\} \hat{\theta}\, d\hat{\Omega}. \tag{11}$$

Assuming that W varies sharply in the neighborhood of $\varphi = -s + S(x,t) = 0$, we have $W_\varphi^2 = \dfrac{\rho^2}{\theta}\omega$ (here ρ and ω are nonzero in small neighborhoods of

$\alpha^i = x^i_{\text{init}}$ and $\varphi = 0$, respectively). Requiring, as a condition on ρ, ω, that $\iiiint \rho^2 \omega \, d\hat{\Omega} = 1^*$, and contracting the region of integration $\hat{\Omega}$ to a point, we obtain that for the singular part of the W-field

$$\mathcal{E} = \frac{(\nabla\varphi)^2}{2m} + U, \qquad \mathcal{P} = \nabla\varphi,$$

which coincides with the values of the energy and momentum possessed by a classical particle.

Moscow State University
named after M. V. Lomonosov

Received
12 X 1962

CITED LITERATURE

1. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 2, N. Y.–London, 1962.

2. Yu. B. Rumer, *Studies in 5-Optics*, Moscow, 1956.

3. B. Finzi, Atti Accad. Naz. Lincei, 6, 18 (1949).

* On $\varphi = -s + S(x,t) = \text{const},$ one may take $x^4 = \lambda + \alpha^4,$ whence $\hat{\theta} = \theta$.

Note: Figure translations are in progress. See original paper for figures.