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Abstract

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ANALYTIC PROPERTIES OF CONTRIBUTIONS OF FEYNMAN DIAGRAMS

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After the first encouraging results achieved in substantiating the Mandelstam representation, thanks to the works of Mandelstam ([1]), Tarski ([2]), and Vladimirov ([3]), it became clear that at present there is no general approach to the solution of this problem even in perturbation theory.

The Landau equations ([4,5]) determine all singularities of the contributions of Feynman diagrams to the scattering amplitude, but on this path serious difficulties arise both in the explicit determination of the singularities of the contributions and in solving the question of on which sheet of the Riemann surface these singularities actually exist.

The singular points of the contributions of Feynman diagrams are located on the so-called singularity surfaces, which, generally speaking, are analytic surfaces. Unfortunately, there is no effective method for determining which points of a singularity surface are singular for the contribution and which are regular. This leads to the fact that even in those cases when the singularity surface has been computed explicitly ([6]), it is impossible to answer the question whether the Mandelstam representation holds.

Therefore any attempt to use the apparatus of the theory of functions of several complex variables for solving this question—central in the Mandelstam problem—about the character of the points of the singularity surface is of interest. In the present work Bremermann' s continuity theorem ([7]) is used for this purpose. Scattering of scalar mesons of equal masses is considered.

1. The contribution to the scattering amplitude in perturbation theory can be represented in the following form

$$F(s,t) = \int_{\widetilde{T}n} \frac{\delta\left(\sum \alpha_i - 1\right) d(\alpha)\, d\alpha_1 \dots d\alpha_n}{\{f(\alpha)s + g(\alpha)t - m^2 k(\alpha)\}^{i}},$$

$$T_n(\alpha_i \geqslant 0,\ i = 1, \dots, n) \mid \alpha = (\alpha_1, \dots, \alpha_n), \tag{1}$$

where $d(\alpha), f(\alpha), g(\alpha), k(\alpha)$ are real homogeneous polynomials, s and t are Mandelstam variables, and m is the meson mass.

Concerning the denominator in the integrand (1) it is known ([8–10]) that $f(\alpha)s+g(\alpha)t-m^2k(\alpha)<0$ for s and t from the region $B\,(s,\ t\,|\,s<4m^2,\ t<4m^2,\ s+t>0)$. If one introduces the variable u, related to the variables s and t by the relation $s+t+u=4m^2$, then the region B is represented as the set of points for which the inequalities $s<4m^2,\ t<4m^2,\ u<4m^2$ hold.

The functions $f(\alpha)$ and $g(\alpha)$ are, generally speaking, sign-changing. For the subsequent arguments it is more convenient to have positive $f(\alpha)$ and $g(\alpha)$ in the denominator of the integrand (1). It turns out that this can always be achieved.

Lemma 1. The function $F(s,t)$ can be represented as a sum of functions of three types: $F_1(s,t), F_2(s,u), F_3(u,t)$; in the denominators of the corresponding formulas for $F_1(s,t), F_2(s,u), F_3(u,t)$, analogous to formula (1), the coefficients at s,t; at s,u; at u,t are positive, and the denominators themselves

are different from zero for $s<4m^2$, $t<4m^2$; for $s<4m^2$, $u<4m^2$; for $u<4m^2$, $t<4m^2$, respectively, for $F_1(s,t)$, $F_2(s,u)$, $F_3(u,t)$.

In accordance with Lemma 1 it is sufficient for us to consider the function $F(s,t)$, defined by the integral (1), in which $f(\alpha)\geqslant 0$, $g(\alpha)\geqslant 0$, and the denominator $f(\alpha)s+g(\alpha)t-m^2k(\alpha)<0$ in the region $B_1(s,t\mid s<4m^2,\ t<4m^2)$.

Since the variables s and t enter the denominator linearly, the function $F(s,t)$ is holomorphic in the tubular domain (7) with base B_1. Since $f(\alpha)\geqslant 0$, $g(\alpha)\geqslant 0$, the domain of holomorphy of the function $F(s,t)$ also includes complex points (s,t) for which $\operatorname{Im} s$ and $\operatorname{Im} t$ have the same sign.

Therefore singular points of the function $F(s,t)$ may arise for $\operatorname{Re} s\geqslant 4m^2$ or $\operatorname{Re} t\geqslant 4m^2$ only for s and t with $\operatorname{Im} s$ and $\operatorname{Im} t$ of opposite signs. We shall consider the function $F(s,t)$ in the region where $\operatorname{Im} s$ and $\operatorname{Im} t$ have opposite signs.

By elementary arguments the following is proved.

Lemma 2. The function $F(s,t)$ is holomorphic in the domain

$$\operatorname{Re} t-4m^2-(\operatorname{Re} s-4m^2)\frac{\operatorname{Im} t}{\operatorname{Im} s}<0. \tag{2}$$

(For Lemma 2 see also (9).)

It follows from Lemma 2 that on the boundary of the domain of holomorphy of the function $F(s,t)$ there is the hypersurface

$$\Phi=(\operatorname{Re} t-4m^2)\operatorname{Im} s-(\operatorname{Re} s-4m^2)\operatorname{Im} t=0. \tag{3}$$

By a simple calculation one can verify that the Levi determinant of this hypersurface is equal to zero.

2. Let us make the basic assumption that the singular points of the function $F(s,t)$ are situated on analytic surfaces $g(s,t)=0$ (for analytic surfaces see (1)). (Here we are making an overly general assumption; for the contributions (1) the function $g(s,t)$ is a polynomial.)

We shall be interested in the question of which points of the analytic surface $g(s,t)=0$ will be singular points of the function $F(s,t)$, and which will be regular. We shall solve this question with the aid of Bremmermann' s continuity theorem (7), which may be formulated as follows.

Let there be a continuous family of analytic planes $E(c)$ in C_2 (the space of two complex variables)

$$as+bt=c, \tag{4}$$

where a,b,c are real numbers, $a^2+b^2=1$. Suppose that on these planes there are certain domains $G(c)$, which converge continuously to the domain $G(c_0)$ on the boundary plane $E(c_0)$. Then from the holomorphy of the function $F(s,t)$ at all points of the approximating domains $G(c)$, and from holomorphy at one single point of the approximated domain $G(c_0)$, there follows the holomorphy of the function $F(s,t)$ throughout the domain $G(c_0)$.

To make the results more transparent, we shall assume that the analytic surface (the surface of singularities) consists of ordinary points, i.e. that it can be represented in the form $t=t(s)$, and that its intersection with the real $(\operatorname{Re}s,\operatorname{Re}t)$-plane for $\operatorname{Re}s\geqslant 4m^2$ or $\operatorname{Re}t\geqslant 4m^2$ forms a curve (the curve of singularities) which is convex and is located in the region $\operatorname{Re}s\geqslant 4m^2$, $\operatorname{Re}t\geqslant 4m^2$.

Using Lemma 2 and the continuity theorem, one can prove the following basic theorem:

Theorem 1. The function $F(s,t)$ is holomorphic for $|\operatorname{Im}s|>0$, $|\operatorname{Im}t|>0$ on the analytic planes $as+bt=c$, $a>0$, $b>0$, where the parameter $c\leqslant c_0$, and the number c_0 is equal to the distance of the tangent to the curve of singularities with slope $-a/b$ from the origin.

Proof. Consider the family of analytic planes $as+bt=c$, or $a\operatorname{Im}s+b\operatorname{Im}t=0$, $a\operatorname{Re}s+b\operatorname{Re}t=c$, and select on them the domain $G_1(c)(s,t\mid \operatorname{Im}s<0,\ \operatorname{Im}t>0)$. From the relation $\operatorname{Im}t/\operatorname{Im}s=-a/b$

and from Lemma 2 we obtain that the function $F(s,t)$ is holomorphic in the domains $G(c)$ on all planes $as+bt=c$ for

$$c<c_1=\frac{4m^2(\operatorname{Im}t-\operatorname{Im}s)}{\sqrt{(\operatorname{Im}t)^2+(\operatorname{Im}s)^2}}=\frac{4m^2(1+b/a)}{\sqrt{1+b^2/a^2}}. \tag{5}$$

We shall prove that also for $c=c_1$ the function $F(s,t)$ is holomorphic in the domain $G_1(c)$. Indeed, the analytic surface $t=t(s)$ and the analytic plane

$as+bt=c$ can have only a finite number of intersection points in a finite domain. Consequently, in the domain $G_1(c_1)$ there are points at which the function $F(s,t)$ is holomorphic. By the continuity theorem ([7]), the function $F(s,t)$ will be holomorphic in the whole domain $G_1(c_1)$. Exactly the same arguments can also be carried out for the domain $G_2(c_1)(s,t \mid \operatorname{Im} s > 0,\ \operatorname{Im} t < 0)$. In what follows we shall reason with the domain $G_1(c)$; in considering the domain $G_2(c)$ no additional difficulties arise.

If $c_1 = c_0$, then the theorem is proved; if $c_1 < c_0$, then we reason as follows. Since c_1 is less than the distance from the origin to the tangent to the curve of singularities, then for $c_1 < c < c_0$ the surface $t = t(s)$ and the plane $as+bt=c$ have intersection points only when $|\operatorname{Im} s| > 0,\ |\operatorname{Im} t| > 0$. Consequently, these intersection points are interior points of the domain $G_1(c)$ or $G_2(c)$, and their distance from the boundary of the domain $G_1(c)$ or $G_2(c)$ is greater than some number $\varepsilon > 0$. We shall show that these intersection points will be regular points of the function $F(s,t)$.

Indeed, in the domain $G_1(c_1)$ the function $F(s,t)$ is holomorphic, and therefore all interior points of the domain $G_1(c_1)$ at a distance $d > \varepsilon$ from the boundary of the domain $G_1(c_1)$ are situated at some minimal positive distance $\delta < \varepsilon$ from the boundary of the domain of holomorphy of the function $F(s,t)$. Hence it follows that in any domain $G_1(c)$ with $c_1 < c < c_1 + \delta$ the function $F(s,t)$ is holomorphic, since the distance of all interior points of the domain $G_1(c)$, including the intersection points of the surface $t=t(s)$ with the plane $as+bt=c$, from the interior points of the domain $G_1(c_1)$ is less than δ. Applying the continuity theorem, we obtain that the function $F(s,t)$ is holomorphic also in the domain $G_1(c_1+\delta)$. Continuing this procedure, we show that the function $F(s,t)$ is holomorphic in $G_1(c)$ for $c \le c_0$. Reasoning in the same way for the domain $G_2(c)$, we obtain that the function $F(s,t)$ is holomorphic in the domain $G_2(c)$ for $c < c_0$. The theorem is proved.

Naturally the question arises whether one can investigate the intersection points of the surface of singularities $t=t(s)$ with the plane $as+bt=c$ also for $c > c_0$. Here a situation may arise in which the real intersection points for $c > c_0$ become complex. Since these points, for some $c > c_0$, will be located on the boundary of the domains $G_1(c)$ or $G_2(c)$ $(\operatorname{Im} s = 0,\ \operatorname{Im} t = 0)$, the arguments of Theorem 1 will not apply.

If, however, for $c > c_0$ all intersection points are real, then the function $F(s,t)$ will be holomorphic for $|\operatorname{Im} s| > 0,\ |\operatorname{Im} t| > 0$ on all planes $as+bt=c$. The function $F(s,t)$ will be holomorphic for $|\operatorname{Im} s| > 0,\ |\operatorname{Im} t| > 0$ on the planes $as+bt=c$ for $c > c_0$ also in the case when among the intersection points there are both complex and real points; one need only require that the real intersection points not become complex.

The results of the theorem can easily be transferred also to the case when the surface of singularities $g(s,t)=0$ is reducible, and to the case when the curve of singularities is not convex.

It should be noted that, in order that in the general case the function $F(s,t)$ have no complex singularities for $|\operatorname{Im} s| > 0,\ |\operatorname{Im} t| > 0$, it is sufficient to require that the function $F(s,t)$ be holomorphic for $\rho_1 > |\operatorname{Im} s| > 0,\ \rho_2 > |\operatorname{Im} t| > 0$ (ρ_1 and ρ_2 are arbitrary numbers), since in this case the possible singularities at the intersection points of the surface of singularities with the plane $as + bt = c$ will always be interior points of the domains $G_1(c)$ or $G_2(c)$, and, applying the continuity theorem, one can show that these points will be regular.

As an example, let us consider the fourth order in perturbation theory (1-3). In this case the singularity surface has the form $st - 4m^2(s+t) + 12m^4 = 0$, and for $c > c_0$ has only two real intersection points with any plane $as + bt = c$. Consequently, at all complex points of the singularity surface $st - 4m^2(s+t) + 12m^4 = 0$ the function $F(s,t)$ is holomorphic.

The results obtained can easily be carried over also to the case of scattering of particles of different masses. The essential point for particles of different masses is to establish the form of the analogue of the domain B. The subsequent proof will be the same as in the case of scattering of particles of equal masses.

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Note: Figure translations are in progress. See original paper for figures.