V. E. LYANTSE

1963

SovietRxiv

Abstract

Full Text

V. E. LYANTSE

ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR WITH SPECTRAL SINGULARITIES

(Presented by Academician I. M. Vinogradov on 5 X 1962)

Let L_θ be the operator generated by the differential expression $l[y] = -y'' + p(x)y$, defined on the half-axis $R^+ = [0, \infty)$, and by the boundary condition $y'(0) - \theta y(0) = 0$; $p(x)$ is a summable complex-valued function, and θ is a complex number. The operator L_θ is considered in the Hilbert space $L^2(R^+)$; its domain of definition $\mathfrak{D}(L_\theta)$ consists of functions $f \in L^2(R^+)$ which have an absolutely continuous derivative f', satisfy the condition $f'(0) = \theta f(0)$, and for which $l[f] = L_\theta f \in L^2(R^+)$. The foundations of the spectral theory of the (non-self-adjoint) operator L_θ were given in the work of M. A. Naimark ([1]).

Let $y_1(x, s)$ be a solution of the differential equation $l[y] = s^2 y$, satisfying the integral equation

$$y_1(x, s) = e^{ixs} - \int_x^\infty \frac{\sin(x-t)}{s} p(t) y_1(t, s)\, dt. \tag{1}$$

Put $A(s) = y'_{1x}(0, s) - \theta y_1(0, s)$. We shall say that the operator L_θ satisfies condition (H) if the equation $A(s) = 0$ has no real roots. Otherwise we shall say that L_θ is an operator with spectral singularities.

In the present article the operator L_θ with spectral singularities is studied. In order to overcome the specific difficulties that arise in this case, we had to impose on the function $p(x)$ the restriction

$$\int_0^\infty e^{\varepsilon x} |p(x)|\, dx < \infty, \tag{2}$$

where $\varepsilon > 0$. In our investigation we rely on the results of M. A. Naimark; moreover, of fundamental importance for us is the integral representation of the resolvent of the operator L_θ with spectral singularities (see ([1]), § 7). The condition (2) itself, however, is also borrowed by us from ([1]).

If condition (2) is satisfied, then the equation $A(s) = 0$ has only a finite number of roots in the half-plane $\operatorname{Im} s \geq 0$. Since we assume that L_θ is an operator with spectral singularities, the equation $A(s) = 0$, in addition to the roots $s_1, \dots, s_r$ located in the half-plane $\operatorname{Im} s > 0$, has also real roots $\sigma_1, \dots, \sigma_\rho$. The numbers $\lambda_1 = s_1^2, \dots, \lambda_r = s_r^2$ are eigenvalues of the operator L_θ. The numbers $\mu_1 = \sigma_1^2, \dots, \mu_\rho = \sigma_\rho^2$ we shall call its spectral singularities. If the number s_k is a root of multiplicity m_k of the equation $A(s) = 0$, then the multiplicity of the eigenvalue λ_k is also equal to m_k, and to this eigenvalue there corresponds a chain of principal functions

$$\{(d/d\lambda)^j \omega(x, \lambda)\}_{\lambda=\lambda_k}, \qquad j = 0, \dots, m_k - 1, \quad k = 1, \dots, r,$$

where $\omega(x, \lambda)$ is the solution of the equation $l[y] = \lambda y$ with the initial values $\omega(0, \lambda) = 1$, $\omega_x(0, \lambda) = 0$. The spectrum Λ_θ of the operator L_θ consists of the eigenvalues $\lambda_1, \dots, \lambda_r$ and points of the continuous spectrum, filling the half-axis $\lambda \geqslant 0$. Thus, the spectral singularities belong to the continuous spectrum. For $\lambda \notin \Lambda_\theta$ the function $\omega(x, \lambda)$ tends exponentially to infinity as $x \to \infty$, and therefore there exist functions $f \in L^2(R^+)$ for which the integral

$$\omega(f, \lambda) = \int_0^\infty f(x) \omega(x, \lambda)\, dx \tag{3}$$

diverges for $\lambda \notin \Lambda_\theta$. The integral (3) exists almost everywhere for $\lambda \geqslant 0$ in the sense of mean-square convergence for every function $f \in L^2(R^+)$. The integrals

$$\omega_j(f, \lambda_k) = \int_0^\infty f(x) \left\{ \left(\frac{d}{d\lambda} \right)^j \omega(x, \lambda) \right\}_{\lambda=\lambda_k} dx, \tag{4}$$

$$j = 0, \dots, m_k - 1; \qquad k = 1, \dots, r,$$

converge in the ordinary sense for every function $f \in L^2(R^+)$. Proofs of all these facts may be found in paper ([1]) (see also ([2])).

Definition 1. A function $\varphi(\lambda)$, measurable for $\lambda > 0$ and holomorphic in a neighborhood of the points $\lambda_1, \dots, \lambda_r$, will be called a **function given on the spectrum** of the operator L_θ. Functions $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$, given on the spectrum of the operator L_θ, will be called **equal** if $\varphi_1(\lambda) = \varphi_2(\lambda)$ almost everywhere for $\lambda > 0$ and

$$\left\{ \left(\frac{d}{d\lambda} \right)^j [\varphi_1(\lambda) - \varphi_2(\lambda)] \right\}_{\lambda=\lambda_k} = 0$$

for $j = 0, \dots, m_k - 1$; $k = 1, \dots, r$. The L_θ-**transform** (Fourier transform) of a function $f(x) \in L^2(R^+)$ will be called the function $\omega f(\lambda)$, given on the spectrum of the operator L_θ by means of the relations $\omega f(\lambda) = \omega(f, \lambda)$ almost everywhere for $\lambda > 0$ and

$$\left\{ \left(\frac{d}{d\lambda}\right)^j \omega f(\lambda) \right\}_{\lambda=\lambda_k} = \omega_j(f, \lambda_k), \qquad j = 0, \dots, m_k - 1; \quad k = 1, \dots, r$$

(see formulas (3) and (4)).

Let us consider the problem of inversion of the L_θ-transform. We note that, using M. A. Naimark' s formula for the resolvent of an operator L_θ with spectral singularities, it is not hard to construct inversion formulas for finite functions $f \in \mathfrak{D}(L_\theta)$. However, the formulas constructed in this way contain integrals over contours that (partly) do not belong to the spectrum of the operator L_θ, and therefore these formulas have no meaning for nonfinite functions for which the integral (3) diverges for $\lambda \notin \Lambda_\theta$. We also note that V. A. Marchenko in paper ([3]) constructed inversion formulas for finite functions under a minimal restriction: assuming only local summability of the "potential" $p(x)$. Below we investigate the question of constructing an inversion formula for nonfinite functions.

Definition. We shall say that a function $\varphi(\lambda)$, given on the spectrum of the operator L_θ, belongs to the class $\mathfrak{G}^\theta$ if it satisfies the condition

$$\int_{-\infty}^{\infty} \left| \frac{\sigma}{A(\sigma)} \varphi(\sigma^2) \right|^2 d\sigma < \infty. \tag{5}$$

By $\mathfrak{H}^\theta$ we denote the manifold of those functions $f \in L^2(R^+)$ for which $\omega f \in \mathfrak{G}^\theta$.

Theorem 1. 1) The manifold $\mathfrak{H}^\theta$ is dense in the space $L^2(R^+)$.

2) The transform $f \to \omega f$ is a one-to-one mapping of the manifold $\mathfrak{H}^\theta$ onto the (entire) class $\mathfrak{G}^\theta$.

3) The inverse transform $\varphi \to \omega^{-1}\varphi = f$ is given by the formula

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\varphi(\lambda)\omega(x,\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})}\, d\lambda + \sum_{k=1}^{r} \left\{ \left(\frac{d}{d\lambda}\right)^{m_k - 1} M_k(\lambda)\varphi(\lambda)\omega(x,\lambda) \right\}_{\lambda=\lambda_k}, \tag{6}$$

where

$$M_k(\lambda) = \frac{(\lambda - \lambda_k)^{m_k} y_1(0, \sqrt{\lambda})}{(m_k - 1)!\, A(\sqrt{\lambda})}.$$

4) For every function $\varphi \in \mathfrak{G}^\theta$ the integral on the right-hand side of formula (6) converges in the sense of $L^2(R^+)$.

5) For any functions $f \in \mathfrak{H}^\theta$ and $g \in L^2(R^+)$ the Parseval equality holds:

$$\int_0^{\infty} f(x)g(x)\,dx = \frac{1}{\pi}\int_0^{\infty} \frac{\omega f(\lambda)\omega g(\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})}\,d\lambda +$$

$$+\sum_{k=1}^{r}\left\{\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\omega f(\lambda)\omega g(\lambda)\right\}_{\lambda=\lambda_k}. \tag{7}$$

Let us note that if the operator L_θ satisfies condition (H), then $\mathfrak{H}^\theta = L^2(R^+)$, and (6) (up to obvious transformations) coincides with the corresponding formula of M. A. Naimark. Theorem 1 asserts, in particular, the completeness of the eigen- and associated elements of the operator L_θ. Indeed, since the set $\mathfrak{H}^\theta$ is dense in $L^2(R^+)$, it follows from equality (7) that every function $g \in L^2(R^+)$ is uniquely determined by its L_θ-transform ωf. Let us also note that the manifold $\mathfrak{H}^\theta$ consists of functions f admitting the representation $f = (1 + \mathfrak{L})\varkappa(D)h$, where $\mathfrak{L}$ is a certain completely continuous Volterra operator, $D = i^{-1}d/dx$, and $\varkappa(s) = (s-\sigma_1)^{n_1}\cdots(s-\sigma_\rho)^{n_\rho}$, where $\sigma_1, \ldots, \sigma_\rho$ are the real roots of the equation $A(s) = 0$, and $n_1, \ldots, n_\rho$ are their multiplicities.

Theorem 2. 1) Let $f \in \mathfrak{H}^\theta$. In order that the function f belong to the domain of definition of the operator L_θ, it is necessary and sufficient that the function $\lambda\omega f(\lambda)$ belong to the class $\mathfrak{S}^\theta$.

2) Let the number z be neither an eigenvalue nor a spectral singularity of the operator L_θ. If $f \in \mathfrak{D}(L_\theta)$ and $(L_\theta - z\cdot 1)f \in \mathfrak{H}^\varkappa$, then $f \in \mathfrak{H}^\theta$. Therefore, for every function $g \in \mathfrak{H}^\theta \cap \mathfrak{D}(R_z)$,

$$(L_\theta - z\cdot 1)^{-1}g(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\omega g(\lambda)\omega(x,\lambda)\sqrt{\lambda}}{(\lambda - z)A(\sqrt{\lambda})A(-\sqrt{\lambda})}\,d\lambda +$$

$$+\sum_{k=1}^{r}\left\{\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\frac{\omega g(\lambda)}{\lambda - z}\omega(x,\lambda)\right\}_{\lambda=\lambda_k}. \tag{8}$$

Let us note that assertion 1) is analogous to the spectral characterization of the domain of definition of a self-adjoint operator.

Denote by $\mathfrak{D}$ the class of Borel subsets of the spectrum of the operator L_θ, each of which is at a positive distance from the spectral singularities. For every set $\Delta \in \mathfrak{D}$ put

$$P(\Delta)f(x) = \tag{9}$$

$$= \frac{1}{\pi}\int_{\Delta\cap(0,\infty)} \frac{\omega f(\lambda)\omega(x,\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})(-\sqrt{\lambda})}\,d\lambda + \sum_{\lambda_k\in\Delta}\left\{\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\omega f(\lambda)\omega(x,\lambda)\right\}_{\lambda=\lambda_k}.$$

Theorem 3. 1) For every set $\Delta \in \mathfrak{D}$, formula (9) defines a linear bounded operator $P(\Delta)$ mapping the whole space $L^2(R^+)$ into itself.

2) The operator function $P : \Delta \to P(\Delta)$ is a generalized spectral measure (see (4)). In particular, the operator function P has the following properties:

a) $P(\Delta_1)P(\Delta_2) = P(\Delta_1 \cap \Delta_2)$ for any $\Delta_1, \Delta_2 \in \mathfrak{D}$;

b) if $\Delta, \Delta_1, \Delta_2, \ldots \in \mathfrak{D}$, $\Delta = \bigcup \Delta_j$, and the sets Δ_j are pairwise nonintersecting, then $P(\Delta) = P(\Delta_1) + P(\Delta_2) + \cdots$, where the series converges in the sense of strong convergence of operators in $L^2(R^+)$;

c) if, for some function $f \in L^2(R^+)$, the equality $P(\Delta)f = 0$ holds for all $\Delta \in \mathfrak{D}$, then $f = 0$;

d) the operator function $\Delta \to [P(\Delta)]^*$ also has property c) (and, of course, properties a) and b)).

3) If the distance from the set Δ to the spectral singularities $\mu_1, \ldots, \mu_\rho$ tends to zero, then $\|P(\Delta)\| \to \infty$.

Obviously, $P(\Delta)$ is a projection operator. It follows from assertion 3) that, as the set Δ approaches the spectral singularities, the “angle of projection” tends to zero. We note that the study of various operators that generate an unbounded (in norm) spectral measure is the subject of a paper by J. Schwartz ([5]).

Theorem 4. The operator L_θ is a generalized spectral operator (see (4)),

$$L_\theta = S + N$$

with scalar part

$$S = \int \lambda P(d\lambda),$$

$$Sf(x) = \frac{1}{\pi}\int_0^\infty \frac{\lambda\omega f(\lambda)\omega(x,\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})}\,d\lambda + \sum_{k=1}^{r}\lambda_k\left\{\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\omega f(\lambda)\omega(x,\lambda)\right\}_{\lambda=\lambda_k},$$

where $f \in \mathfrak{H}^\theta$, and with root part N, which is a bounded nilpotent operator,

$$Nf(x) = \sum_{k=1}^{r}(m_k - 1)\left\{\left(\frac{d}{d\lambda}\right)^{m_k-2} M_k(\lambda)\omega f(\lambda)\omega(x,\lambda)\right\}_{\lambda=\lambda_k}, \qquad f \in L^2(R^+).$$

In particular, for each set $\Delta \in \mathfrak{D}$ the spectrum of the restriction of the operator L_θ to the invariant subspace $P(\Delta)L^2(R^+)$ is contained in the set $\overline{\Delta}$. For each bounded set $\Delta \in \mathfrak{D}$ there is the inclusion

$$P(\Delta)L^2(R^+) \subset \mathfrak{D}(L_\theta) \cap \mathfrak{H}^\theta.$$

It is known that if the operator L_θ satisfies condition (H), then it is a spectral operator in the usual sense (see, for example, the survey ([6])). The essential difference between an operator L_θ satisfying condition (H) and an operator L_θ with spectral singularities is as follows.

The class $\mathfrak{G}^\theta$ of functions $\varphi(\lambda)$, defined on the spectrum of the operator and satisfying condition (5), can be turned into a Hilbert space by setting

$$\|\varphi\|^2 = \int_{-\infty}^{\infty} \left| \frac{\sigma}{A(\sigma)} \varphi(\sigma^2) \right|^2 d\sigma + \sum_{k=1}^{r} \sum_{j=0}^{m_k - 1} \frac{1}{j!} \left| \left\{ \left(\frac{d}{d\lambda} \right)^j \varphi(\lambda) \right\}_{\lambda=\lambda_k} \right|^2 .$$

In the first case, the L_θ-transform is a one-to-one and continuous mapping of the Hilbert space $L^2(R^+)$ onto the Hilbert space $\mathfrak{G}^\theta$, under which the operator L_θ corresponds to multiplication by λ. In the second case, the image of the space $L^2(R^+)$ under the L_θ-transform is larger than the space $\mathfrak{G}^\theta$; however, for each set $\Delta \in \mathfrak{D}$, the mapping

$$\omega : P(\Delta)L^2(R^+) \to \mathfrak{G}^\theta$$

is continuous, and the linear hull of the subspaces $P(\Delta)L^2(R^+)$, $\Delta \in \mathfrak{D}$, is dense in $L^2(R^+)$.

Lviv Polytechnic Institute

Received
3 X 1962

CITED LITERATURE

1. M. A. Naimark, *Tr. Mosk. matem. obshch.*, **3**, 181 (1954).
2. B. Ya. Levin, DAN, **106**, No. 2 (1956).
3. V. A. Marchenko, *Matem. sborn.*, **52** (94), No. 2 (1960).
4. V. E. Lyantse, DAN, **142**, No. 2 (1962).
5. J. Schwartz, *Comm. Pure and Appl. Math.*, **13**, No. 3, 509 (1960).

6. N. Dunford, *Sborn. per.*, *Matematika*, **4**, 1, 1960.

Note: Figure translations are in progress. See original paper for figures.