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Abstract

Full Text

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Some Embedding Theorems in Orlicz Classes

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In this paper some embedding theorems in Orlicz classes or spaces are obtained. In particular, Theorems 1 and 2 strengthen the well-known result of S. L. Sobolev on the embedding of the space $W_p^{(l)}$ $(pl = n)$ into $\mathcal{L}_q$ (with $q \geqslant 1$ arbitrary). Next, the embedding of Orlicz classes into the space of continuous functions is considered. Throughout our paper one may consider the functions $M(u)$ forming the Orlicz classes to grow faster than any power function. Embedding theorems for functions $M(u)$ growing no faster than a certain power function were considered, for example, in ([5],[6]).

Notation: R_n is n-dimensional Euclidean space; $x = (x_1, \dots, x_n)$; G is a domain in the space R_n satisfying the cone condition (see (1)); $W_p^{(l)}(G)$ is the space of functions $u(x)$ having generalized derivatives up to order l, summable with power p:

$$\|D^j u\|_p^p = \sum_{i_1+\dots+i_n=j} \int_G \left| \frac{\partial^j u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \right|^p dx, \qquad \|u\|^p_{W_p^{(l)}} = \sum_{j=0}^{l} \|D^j u\|_p^p;$$

$\mathring{W}_p^{(l)}(G)$ is the subspace of $W_p^{(l)}(G)$ whose elements are functions $u(x)$ vanishing on the average on the boundary of the domain G, together with their derivatives up to order $l-1$.

Further, $M(u)$ is an N-function, $N(v)$ is the N-function complementary to $M(u)$; $\mathcal{L}_M(G)$ and $\mathcal{L}_M^*(G)$ are the Orlicz class and space with norm

$$\|u\|_M = \sup_v \left| \int_G uv\, dx \right| \quad \left(\int_G N(v)\, dx \leqslant 1 \right);$$

E_M is the closure of the bounded functions in the norm $\|u\|_M$. It is known that $E_M \subset \mathcal{L}_M \subset \mathcal{L}_M^*$. Further necessary information on the theory of Orlicz classes may be found, for example, in ([4]).

§ 1. Embedding of $\mathring{W}_p^{(l)}$ $(pl = n)$ into the Orlicz class $\mathcal{L}_M$

Let

$$M(u) \geqslant 0$$

be an entire N-function, i.e.

$$M(u) = \sum_{q \geqslant 2} a_q |u|^q,$$

and let the series converge on the whole line $-\infty < u < +\infty$. This means that

$$\lim_{q \to \infty} |a_q|^{1/q} = 0.$$

Assume that also

$$\lim_{q \to \infty} |a_q|^{1/q} q = 0,$$

and put

$$M_1(u) = \operatorname{mes} G \cdot \sum |a_q| C_q^q |u|^q, \quad \text{where } C_q = \left(\frac{n-1}{2}\right)^l \frac{q^l}{\prod\limits_{m=1}^{l} (n + (l-m)q)}.$$

Obviously, $M_1(u)$ is an entire function.

Theorem 1. Let $u(x) \in \overset{\circ}{W}{}_p^{(l)}$ $(pl = n)$. Then $u(x) \in \mathcal{L}_M$ (even E_M), and the inequality

$$\int_G M(u)\, dx \leqslant M_1(\|D^l u\|_p). \tag{1}$$

holds.

The embedding operator is completely continuous in the norm $\|u\|_M$.

Proof. For the proof we cite two known inequalities (see, for example, ([2,3])):

$$\|u\|_{p_0} \leq (\operatorname{mes} G)^{\frac{p_1 - p_0}{p_0 p_1}} \|u\|_{p_1} \quad (p_0 \leq p_1); \tag{2}$$

$$\|u\|_{\frac{np}{n-p}} \leq \frac{np - p}{2(n-p)} \|Du\|_p, \tag{3}$$

where $u \in \overset{0}{W}{}_p^{(1)}(G)$ or $u \in W_p^{(1)}(R_n)$.

Applying (3) successively, we obtain the inequality

$$\|u\|_{\frac{np}{n-pl}} \leq \left(\frac{n-1}{2}\right)^l \frac{p^l}{\prod\limits_{m=1}^{l} (n - mp)} \|D^l u\|_p, \tag{4}$$

whence, putting $\dfrac{np}{n-pl} = q$, we shall have

$$\|u\|_q \leq \left(\frac{n-1}{2}\right)^l \frac{q^l}{\prod\limits_{m=1}^{l}(n+(l-m)q)} \|D^l u\|_{\frac{nq}{n+lq}} \equiv C_q \|D^l u\|_{\frac{nq}{n+lq}}.$$

Further, since $nq/(n+lq) < n/l = p$, it follows from inequality (2) that

$$\|D^l u\|_{\frac{nq}{n+lq}} \leq (\operatorname{mes} G)^{1/q} \|D^l u\|_p.$$

After this, (1) follows from the chain of inequalities:

$$\int_G M(u)\,dx \leq \sum |a_q| \int_G |u|^q\,dx \leq \sum |a_q| C_q^q \|D^l u\|^q_{\frac{nq}{n+lq}} \leq$$

$$\leq \operatorname{mes} G \sum |a_q| C_q^q \|D^l u\|_p^q \equiv M_1(\|D^l u\|_p).$$

Thus, we obtain that $u \in \mathcal{L}_M$. Moreover, from the conditions of the theorem it is clear that $\lambda u \in \mathcal{L}_M$, where λ is any number. This means that $u \in E_M$. It remains to prove that the set of functions $u(x)$, bounded in $\overset{0}{W}{}_p^{(l)}$ $(pl = n)$, will be compact in the norm of the Orlicz space $\mathcal{L}^*_M$.

We shall use the compactness criterion indicated in (4), according to which a bounded set of functions $u \in E_M$ will be compact if it has equiabsolutely continuous norms and is compact in some other Orlicz space. The second condition is fulfilled by virtue of S. L. Sobolev' s theorem on the embedding of $\overset{0}{W}{}_p^{(l)}$ $(pl = n)$ in $\mathcal{L}_q$ $(q \geq 1)$.

Let us show that the first condition is also fulfilled. For this purpose choose an unbounded increasing sequence of numbers $b_q \geq 0$ such that

$$\lim_{q\to\infty} |a_q|^{1/q} q b_q = 0,$$

and consider the function $M'(u) = \sum |a_q| b_q^q |u|^q$. The function $M'(u)$ satisfies the conditions of Theorem 1 and, consequently,

$$\|u\|_{M'} \leq \int_G M'(u)\,dx + 1 \leq M'_1(\|D^l u\|_p) + 1.$$

Moreover, $M'(u)$ grows as $|u| \to \infty$ substantially faster than $M(u)$, i.e.

$$\lim_{|u|\to\infty} \frac{M(\lambda u)}{M'(u)} = 0$$

for any λ. This means that the norms $\|u\|_M$ are equiabsolutely continuous (see ([4])). Theorem 1 is completely proved.

§ 2. **Embedding of W_p^l $(pl = n)$ in $\mathcal{L}_M$.** First we shall establish

Lemma. *Let $u(x) \in E_M(G)$, and let $x_i'(i = 1, \dots, n)$ be a new orthogonal or affine basis. Denote the function $u(x)$ in this basis by $u'(x')$. Then the expressions $\|u(x)\|_M(G)$, $\|u'(x')\|_M(G')$ are equivalent.*

The proof of this lemma is carried out, as in the paper of E. Gagliardo (*), taking into account the formula

$$\|u\|_M = \inf_k \frac{1}{k} \left(\int_G M[ku(x)]\,dx + 1 \right).$$

As was shown in the same paper, every domain G satisfying the cone condition can be represented in the form of a finite sum (generally speaking, with mutual overlap) of domains G_k $(k = 1, \dots, N)$, each of which is obtained by a "plane-parallel" motion of an n-dimensional parallelepiped. For our purposes it is sufficient to restrict ourselves to the domain G_k and, by virtue of the lemma, to regard the parallelepiped as a cube with edge length d.

Let $M(u)$ be the same as in Theorem 1,

$$M_1(u) = \operatorname{mes} G \sum a_q C_q^q |u|^q,$$

where

$$C_q = C_l^{[n/2]} \frac{[\sigma(n-1)q]^l}{\prod\limits_{m=1}^{l} (n + (l-m)q)}, \qquad \sigma = \max\left(\frac{2e}{d}, 2\right).$$

Theorem 2. Let $u(x) \in W_p^{(l)}(G)$, where $pl = n$. Then $u(x) \in \mathcal{L}_M(G)$ (even $E_M(G)$) and the inequality

$$\int_G M(u)\,dx \le M_1(\|u\|_{W_p^{(l)}}) \tag{5}$$

holds. Moreover, the embedding operator is completely continuous with respect to the norm $\|u\|_M$.

Proof. Using the method of the paper (*), one can obtain the inequality (for the domain G_k)

$$\|D^i u\|_{\frac{np}{n-p}} \leq \sigma \frac{np-p}{n-p}(\|D^i u\|_p + \|D^{i+1} u\|_p). \tag{6}$$

Applying (6) successively, we shall have

$$\|u\|_{\frac{np}{n-pl}} \leq C_n^{[n/2]} \frac{[\sigma(n-1)p]^l}{\prod_{m=1}^{l}(n-mp)} \|u\|_{W_p^{(l)}}. \tag{7}$$

The subsequent arguments are the same as in Theorem 1.

Remark. In Theorem 2 an embedding of the space $W_p^{(l)}$ $(pl = n)$ into Orlicz classes as a whole was obtained. If, however, one restricts oneself to considering the ball $\|u\|_{W_p^{(l)}} \leq R$, then the inequality

$$\int_G M(u)\, dx \leq M_1(\|u\|_{W_p^{(l)}}), \tag{5'}$$

holds, where $M(u) = \sum a_q |u|^q$ is such that

$$\lim_{q\to\infty} |a_q|^{1/q} C_q < 1/R.$$

In particular, taking Stirling' s formula into account, we obtain, for $\alpha < \alpha(R)$,

$$\int_G e^{\alpha|u|}\, dx \leq M_1(\|u\|_{W_p^{(l)}}), \qquad \text{where} \qquad M_1(u) = \operatorname{mes} G \sum \frac{C_q^q \alpha^q}{q!}|u|^q. \tag{5''}$$

§ 3. **Embedding of an Orlicz space into spaces of continuous functions.** We shall make an additional assumption concerning the domain G. Namely, suppose that the domain satisfies the "strong" cone condition. This means that there exist numbers ρ and λ such that for any points P and Q, the distance between which $|P-Q| \leq \rho$, there are two straight circular cones with vertices at these points, and the volume V of their intersection S, lying in G, satisfies the inequality $V \geq \lambda |P-Q|^n$. Next, denote $r = |x|$ and $r_1 = |P-R|$, where R is an arbitrary point of G or R_n. Finally, let $\chi(S)$ be the characteristic function of S.

Theorem 3. Let G satisfy the "strong" cone condition or be all of R_n. Further, let $du/dx_i \in \mathcal{L}_M^*(G)$ $(i = 1, \dots, n)$, $\|r^{1-n}\chi(S_1)\|_N < \infty$, where S_1 is the ball of radius 1.

Then $u(x)$ is a continuous function and for any points P and Q such that $|P - Q| \le \rho$, the inequality holds

$$|u(P) - u(Q)| \le C_1 \|r_1^{1-n}\chi(S)\|_N \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_M . \tag{8}$$

The proof of inequality (8) is carried out by the same method as in work (3). From (8) there immediately follows the equicontinuity of the set of functions for which the $\|\partial u/\partial x_i\|_M$ are uniformly bounded.

Corollary 1. If the function $M(u)$ grows faster than any power function and the domain G is bounded, then $u(x)$ satisfies a Hölder condition with any exponent $\alpha < 1$. Moreover, the inequality holds

$$|u(P) - u(Q)| \le C_1(\alpha)|P - Q|^\alpha \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_M . \tag{9}$$

Indeed, in this case the complementary function $N(v)$ grows more slowly than any function $|v|^{1+\varepsilon}$, and therefore

$$\|r_1^{1-n}\chi(S)\|_N \le C_1(\varepsilon)\|r_1^{1-n}\chi(S)\|_{1+\varepsilon}.$$

Using (8), we obtain

$$\begin{aligned} |u(P) - u(Q)| &\le C_1(\varepsilon)\|r_1^{1-n}\chi(S)\|_{1+\varepsilon} \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_M \\ &= C_1(\varepsilon)\left(\int_S r_1^{(1-n)(1+\varepsilon)}\,dx\right)^{1/(1+\varepsilon)} \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_M \le \\ &\le C_1(\alpha)|P-Q|^\alpha \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_M \qquad \left(\text{for } \varepsilon = \frac{\alpha - 1}{1 - n - \alpha}\right). \end{aligned}$$

It is then not difficult to show that, in the case of a bounded domain, P and Q in (9) may be taken arbitrary; this is what was required to prove.

Corollary 2. If $u(x) \in W_p^{(l+1)}$ $(pl = n)$, then for $|P - Q| \le \rho_1$ (ρ_1 sufficiently small) the inequality holds

$$|u(P) - u(Q)| \le C_2\left(\|u\|_{W_p^{(l+1)}}\right)|P - Q| \ln |P - Q|^{-1}. \tag{10}$$

Indeed, using the remark, in inequality (8) one may take $M(u) = e^{\alpha|u|} - \alpha|u| - 1$. After this, (10) follows from the estimate of $\|r_1^{1-n}\chi(S)\|_N$.

Let us give an example. Let $M(u) \sim |u|^n \ln^\gamma |u|$. Then, as is known (4),

$$N(v) \sim |v|^{\frac{n}{n-1}} \ln^{-\frac{\gamma}{n-1}} |v|.$$

It is not difficult to see that $\|r_1^{1-n}\chi(S_1)\|_N < \infty$, if $\gamma > n-1$. For $|P-Q| \leq \rho_1$ the estimate of the modulus of continuity is given by the inequality

$$|u(P)-u(Q)| \leq C_3 |\ln|P-Q||^{\frac{n-\gamma-1}{n}} \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_M .$$

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Note: Figure translations are in progress. See original paper for figures.