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Abstract

Full Text

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ON SPECTRAL EXPANSIONS OF SYMMETRIC OPERATORS

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1. Consider a closed symmetric operator A, acting in a Hilbert space H and having a domain of definition D_A dense in H. The operator A is not assumed to be self-adjoint, so that A is a part of the operator A^* adjoint to it, and in general the domain of definition D_{A^*} of the operator A^* is wider than D_A. As is known, there exists a resolution of the identity E_λ $(-\infty < \lambda < +\infty)$ such that, for any finite interval $\Delta = [\mu_1, \mu_2)$ and any $f \in H$, the relations* hold:

$$E(\Delta)f \in D_{A^*}, \qquad A^* E(\Delta) f = \int_{\mu_1}^{\mu_2} \lambda \, dE_\lambda f, \tag{1}$$

where $E(\Delta) = E_{\mu_2} - E_{\mu_1}$. This resolution of the identity is, in general, not orthogonal and not unique. Each such resolution of the identity is called a spectral function of the operator A.** According to a well-known theorem of M. A. Naimark ([4]; see also ([1], pp. 369–372)), for every spectral function E_λ $(-\infty < \lambda < +\infty)$ of the operator A there exists, in some Hilbert space $\widehat{H} \supset H$, a self-adjoint extension $\widetilde{A}$ of the operator A such that the orthogonal spectral function $\widetilde{E}_\lambda$ $(-\infty < \lambda < +\infty)$ of the operator $\widetilde{A}$ is related to E_λ by the formula

$$E_\lambda f = P \widetilde{E}_\lambda f \qquad (f \in H), \tag{2}$$

where P is the projection operator in $\widehat{H}$ onto H.

Taking (1) into account, one may regard the equality

$$f = \int_{-\infty}^{+\infty} dE_\lambda f \tag{3}$$

as a certain formula for expansion in generalized eigen-elements of the operator A^*.

Let, for example, A be a symmetric ordinary differential operator with minimal domain of definition in $\mathcal{L}^2(a,b)$, generated by a differential expression $l[y]$ of order $2n$; the endpoints of the interval (a,b) are not assumed to be regular. The spectral functions of such an operator are described in ([5]). In this case formula (3) is realized in the form of an expansion in solutions of the equation

$$l[y] - \lambda y = 0, \tag{4}$$

which also play the role of generalized eigen-elements of the operator A^*. However, if at least one of the endpoints of the interval (a, b) is regular, then, as has been shown in a number of papers ([6–11]), there is a broad class of spectral expansions such that in each of them not all solutions of equation (4) take part,

* The operator function E_λ $(-\infty < \lambda < +\infty)$ is assumed to be continuous from the left. On the definition of a resolution of the identity see, for example, ([1]), pp. 359, 360.

** Such spectral functions were considered for the first time by Carleman ([2]) for symmetric integral operators and by Stone ([3]) for operators in an abstract Hilbert space. The definition of a spectral function given here is due to M. A. Naimark ([4]).

only those which satisfy certain boundary conditions depending on λ.

In the present paper we consider the spectral functions E_λ $(-\infty < \lambda < +\infty)$ of a symmetric operator A acting in an abstract Hilbert space. It turns out that in this general case as well there are "boundary conditions," depending on the parameter λ, which are satisfied, in a certain sense, by the generalized eigenvectors of the operator A^* that participate in the expansion (3).

2. Let E_λ $(-\infty < \lambda < +\infty)$ be some resolution of the identity in $\widetilde{H}$, and let α and β be fixed real numbers. Denote by $K_0(E_\lambda; \alpha, \beta)$ the linear manifold of vector functions $g(\lambda)$ $(\alpha \leq \lambda \leq \beta)$ with values in H, admitting a representation

$$g(\lambda) = \varphi_1(\lambda) g_1 + \varphi_2(\lambda) g_2 + \cdots + \varphi_m(\lambda) g_m,$$

where $\varphi_k(\lambda)$ $(k = 1, 2, \ldots, m)$ are arbitrary continuous complex-valued functions of the parameter λ, and g_k $(k = 1, 2, \ldots, m)$ are arbitrary elements of H; for different vector functions these elements and their number may be different.*

For any vector functions $g(\lambda), h(\lambda) \in K_0(E_\lambda; \alpha, \beta)$ there exists the integral

$$\int_\alpha^\beta ((dE_\lambda) g(\lambda), h(\lambda)) = \lim_{\omega \to 0} \sum_{k=0}^{n-1} ((E_{\lambda_{k+1}} - E_{\lambda_k}) g(\lambda'_k), h(\lambda'_k)), \tag{5}$$

where

$$\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \beta, \qquad \lambda'_k \in [\lambda_k, \lambda_{k+1}] \ (k = 0, 1, \ldots, n-1);$$

$$\omega = \max(\lambda_{k+1} - \lambda_k).$$

Let us introduce the aggregate $K_1(E_\lambda; \alpha, \beta)$ of vector functions $g(\lambda)$ $(\alpha \leq \lambda \leq \beta)$ with values in H satisfying the following condition: for every $\varepsilon > 0$ there exists, for the function $g(\lambda)$, a function $g_\varepsilon(\lambda) \in K_0(E_\lambda; \alpha, \beta)$ such that

$$\overline{\lim_{\omega \to 0}} \sum_{k=0}^{n-1} ((E_{\lambda_{k+1}} - E_{\lambda_k})(g(\lambda'_k) - g_\varepsilon(\lambda'_k)), g(\lambda'_k) - g_\varepsilon(\lambda'_k)) < \varepsilon,$$

where λ_k, λ'_k, and ω have the same meaning as before.

$K_1(E_\lambda; \alpha, \beta)$ is, obviously, a linear manifold, and together with a vector function $g(\lambda)$ it also contains $\varphi(\lambda) g(\lambda)$, whatever continuous complex-valued function $\varphi(\lambda)$ may be. It is easy to see that for any $g(\lambda)$ and $h(\lambda)$ from $K_1(E_\lambda; \alpha, \beta)$ the integral (5) exists.**

Lemma. *If E_μ $(-\infty < \lambda < +\infty)$ is a spectral function of a symmetric operator A, then for any vector functions $g(\lambda)$ and $h(\lambda)$ from $K_1(E_\lambda; \alpha, \beta)$ there exists the integral*

$$\int_\alpha^\beta (A^*(dE_\lambda) g(\lambda), h(\lambda)) = \lim_{\omega \to 0} \sum_{k=0}^{n-1} (A^*(E_{\lambda_{k+1}} - E_{\lambda_k}) g(\lambda'_k), h(\lambda'_k))$$

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* In an analogous way one can define the linear manifold $K_0(E_\lambda; -\infty, +\infty)$, requiring additionally that the functions $\varphi_k(\lambda)$ be finite.

** If the value of the integral (5) is taken as the scalar product $[g, h]$ of the vector functions $g(\lambda)$ and $h(\lambda)$, then $K_1(E_\lambda; \alpha, \beta)$ becomes a Hilbert space, in general incomplete. The completion of the space $K_1(E_\lambda; \alpha, \beta)$ is, obviously, also a completion for $K_0(E_\lambda; \alpha, \beta)$ in this same metric. We note that the completion of the space $K_0(E_\lambda; -\infty, +\infty)$ essentially coincides with the space $\widetilde{H}$ discussed in M. A. Naimark' s theorem (see (1), pp. 363-366).

and the formula holds

$$\int_\alpha^\beta (A^*(dE_\lambda) g(\lambda),\, h(\lambda)) = \int_\alpha^\beta ((dE_\lambda) \lambda g(\lambda),\, h(\lambda)) = \int_\alpha^\beta ((dE_\lambda) g(\lambda),\, \lambda h(\lambda)).$$

3. Let A be a closed symmetric operator in H, let $\widetilde{A}$ be a self-adjoint operator in $\widetilde{H} \supset H$ which is an extension of the operator A, and let E_λ $(-\infty < \lambda < +\infty)$ be the spectral function of the operator A associated with the spectral function $\widetilde{E}_\lambda$ of the operator $\widetilde{A}$ by formula (2). For any complex λ denote by $\widetilde{\mathcal{L}}_\lambda$ the manifold of all $\tilde{f} \in D_{\widetilde{A}}$ such that $\widetilde{A}\tilde{f} - \lambda \tilde{f} \in H$. Put $\mathcal{L}_\lambda = P\widetilde{\mathcal{L}}_\lambda$, where P is the projection operator in $\widetilde{H}$ onto H. It is easy to see that $D_A \subset \mathcal{L}_\lambda \subset D_{A^*}$. We note that the manifold $\mathcal{L}_\lambda$, for any complex λ, is determined uniquely by specifying the spectral function E_λ $(-\infty < \lambda < +\infty)$ of the operator A; here it is not necessary to assume that the operator $\widetilde{A}$ is given.

Theorem 1. Let $g(\lambda)$ $(\alpha \leqslant \lambda \leqslant \beta)$ be a vector function with values in H, satisfying the following conditions: 1) for every $\lambda \in [\alpha, \beta]$, $g(\lambda) \in \mathcal{L}_\lambda$; 2) the vector functions $g(\lambda)$ and $A^* g(\lambda)$ belong to the manifold $K_1(E_\lambda; \alpha, \beta)$; 3) for the vector function $h(\lambda) = A^* g(\lambda) - \lambda g(\lambda)$, for every $\mu \in [\alpha, \beta]$ the equality

$$\lim_{\eta \to 0} \left[\frac{1}{\eta^2} \int_{\mu}^{\mu+\eta} ((dE_\lambda)(h(\lambda) - h(\mu)),\, h(\lambda) - h(\mu)) \right] = 0$$

holds.

Then, for any $f \in H$ and any $\mu_1, \mu_2 \in [\alpha, \beta]$, the equality

$$\int_{\mu_1}^{\mu_2} [(A^*(dE_\lambda) f,\, g(\lambda)) - ((dE_\lambda) f,\, A^* g(\lambda))] = 0. \tag{6}$$

Without dwelling on the proof of the theorem, we shall only note that we carry it out approximately in the same way as M. G. Krein justifies his method of directing functionals (([12]; see also ([1]), pp. 448–454).

Corollary. Let the vector function $g(\lambda)$ $(\alpha \leqslant \lambda \leqslant \beta)$ satisfy condition 1) of Theorem 1 and have the form

$$g(\lambda) = \varphi_1(\lambda) g_1 + \varphi_2(\lambda) g_2 + \dots + \varphi_m(\lambda) g_m,$$

where $g_k \in D_{A^*}$, and $\varphi_k(\lambda)$ are complex-valued functions satisfying the Lipschitz condition $(k = 1, 2, \dots, m)$. Then equality (6) holds.

Denote by $\mathfrak{N}_z$, for any nonreal z, the defect subspace of the operator A, consisting of all $u \in D_{A^*}$ satisfying the equation $A^* u = \overline{z}\, u$. Fix some nonreal value z_0. As shown in ([8], [13]), for any nonreal z from the half-plane $(\operatorname{Im} z \cdot \operatorname{Im} z_0 > 0)$ the manifold $\mathcal{L}_z$ defined above is representable in the form

$$\mathcal{L}_z = D_A + [F(z) - E] \mathfrak{N}_{z_0},$$

where $F(z)$ is some linear operator from $\mathfrak{N}_{z_0}$ into $\mathfrak{N}_{\overline{z}_0}$, satisfying the conditions: a) $\|F(z)\| \leqslant 1$; b) $F(z)$ is an analytic operator function of z in the half-plane $(\operatorname{Im} z \cdot \operatorname{Im} z_0 > 0)$. The correspondence between the collection of all such operator functions $F(z)$ and the set of all spectral functions E_λ $(-\infty < \lambda < +\infty)$ is one-to-one.

Theorem 2. Let $g(\lambda)$ $(\alpha \leqslant \lambda \leqslant \beta)$ be a vector function with values in H satisfying the following conditions: 1) $g(\lambda)$ is the continuous continuation of the vector function $g(z) = f_0 + F(z) u - u$ from the half-plane $(\operatorname{Im} z \cdot \operatorname{Im} z_0 > 0)$, where $f_0 \in D_A$, $u \in \mathfrak{N}_{z_0}$; $\operatorname{Im}(A^* g(\lambda),\, g(\lambda)) = 0$ for every $\lambda \in [\alpha, \beta]$. Then equality (6) holds.

4. Let us explain the meaning of the theorem by the example of a symmetric differential operator A with minimal domain of definition in $\mathcal{L}^2(0, +\infty)$, which is generated by an ordinary differential expression $l[y]$ of arbitrary

even order $2n$. For any functions $y = y(x)$ and $v = v(x)$ from D_{A^*} the equality

$$(A^*y, v) - (y, A^*v) = [y, v]_0^{+\infty}, \tag{7}$$

holds, where $[y, v]$ is a known bilinear form in the functions $y(x)$, $v(x)$ and their quasi-derivatives up to order $(2n-1)$, inclusive. If $E_\lambda(-\infty < \lambda < +\infty)$ is the spectral function of the operator A, then for any real μ and any function $f(x) \in \mathcal{L}^2(0, +\infty)$ the formula

$$(E_\mu - E_0)f = \int_0^\mu y(x, \lambda, f)\, d\rho(\lambda), \tag{8}$$

holds, where $\rho(\lambda)$ is a certain scalar distribution function, and $y(x, \lambda, f)$ is a certain solution of equation (4), depending linearly on f^*. Let $g(\lambda)$ be a vector function of λ $(\alpha \le \lambda \le \beta)$ with values in $H = \mathcal{L}^2(0, +\infty)$, such that $g(\lambda) = g(x, \lambda)$, where $g(x, \lambda)$ is a scalar function of x and λ $(0 \le x < +\infty;\ \alpha \le \lambda \le \beta)$. Suppose that $g(\lambda)$ satisfies all the conditions of Theorem 1 or 2 and that for every $\lambda \in [\alpha, \beta]$, $g(x, \lambda)$ is a finite function of x. Taking (6), (7), and (8) into account, we conclude that for any $\mu_1, \mu_2 \in [\alpha, \beta]$ the equality

$$\int_{\mu_1}^{\mu_2} [\, y(x, \lambda, f), g(x, \lambda)\,]_{x=0}\, d\rho(\lambda) = 0$$

holds. Hence it follows that almost everywhere on the interval $[\alpha, \beta]$ with respect to the measure $\rho(\lambda)$,

$$[\, y(x, \lambda, f), g(x, \lambda)\,]_{x=0} = 0,$$

i.e., $y(x, \lambda, f)$ satisfies a certain boundary condition depending on λ.

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References

1. N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, M.–L., 1950.

2. T. Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala, 1923.

3. M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, N. Y., 1932.

4. M. A. Naimark, Izv. Akad. Nauk SSSR, Ser. Mat., 4, No. 3, 277 (1940).

5. A. V. Shtraus, Izv. Akad. Nauk SSSR, Ser. Mat., 21, No. 6, 785 (1957).

6. A. V. Shtraus, Izv. Akad. Nauk SSSR, Ser. Mat., 20, No. 6, 783 (1956).

7. A. V. Shtraus, DAN, 115, No. 1, 67 (1957).

8. A. V. Shtraus, DAN, 139, No. 2, 316 (1961).

9. A. V. Shtraus, in *Trudy 4th Scientific Conf. of Higher Educational Institutions of the Volga Region*, Kuibyshev, 1961, p. 110.

10. R. A. Oganyan, Dokl. AN ArmSSR, 34, No. 2, 57 (1962).

11. I. S. Kats, DAN, 145, No. 3, 510 (1962).

12. M. G. Krein, *Collection of Papers of the Institute of Mathematics, Academy of Sciences of the Ukrainian SSR*, No. 10, 83 (1948).

13. A. V. Shtraus, Izv. Akad. Nauk SSSR, Ser. Mat., 18, No. 1, 51 (1954).

14. E. A. Coddington, Ann. of Math., 68, No. 2, 378 (1958).

* One may set $\rho(\lambda) = \mathrm{Sp}\, T(\lambda - 0)$, where $T(\lambda)$ is the matrix spectral distribution function (see (5), formulas (5.2) and (5.11)). We note that, without introducing any new means, the results established in (5) can be extended to symmetric ordinary differential operators of arbitrary order with complex coefficients (cf. (14)).

Note: Figure translations are in progress. See original paper for figures.