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Abstract

Full Text

PHYSICS

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ONE GENERALIZATION OF EINSTEIN' S GRAVITATION EQUATIONS

(Presented by Academician N. N. Bogolyubov, 28 VII 1962)

The fundamental equation of Einstein' s general theory of relativity has, as is known, the form

$$R_i^k - \frac{1}{2}\delta_i^k R = \chi T_i^k, \tag{1}$$

where $\chi = 8\pi G/c^4$, $G = \frac{2}{3} \cdot 10^{-7} \text{ cm}^3/\text{g} \cdot \text{sec}^2$ is the gravitational constant;

$$T_i^k = (P + \varepsilon)u_i u^k + \delta_i^k P + \bar{T}_i^k; \tag{2}$$

$$\bar{T}_i^k = \frac{1}{4\pi} \left[F_{il} F^{kl} - \frac{1}{4} F_{lm} F^{lm} \delta_i^k \right]; \tag{3}$$

F_{ik} are the components of the electromagnetic-field tensor; \bar{T}_i^k is the energy-momentum tensor of the electromagnetic field;

$$R_i^k = g^{kr} R_{ir} = g^{kr} \left(\frac{\partial \Gamma_{ir}^l}{\partial x^l} - \frac{\partial \Gamma_{il}^r}{\partial x^r} + \Gamma_{ir}^l \Gamma_{lm}^m - \Gamma_{il}^m \Gamma_{rm}^l \right) \tag{4}$$

is the contracted curvature tensor (the Ricci tensor), where the Christoffel symbols are expressed in terms of the metric tensor

$$\Gamma_{kl}^i = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{5}$$

Since the Bianchi identity holds:

$$R_{ikl;m}^n + R_{imk;l}^n + R_{ilm;k}^n = 0, \tag{6}$$

then, multiplying the terms of this identity by $g^{ik} \delta_n^l$, we obtain the expression $\partial R / \partial x^m - 2R_{m,n}^n = 0$, which is analogous to the identity

$$R_{i;k}^k - \frac{1}{2} \frac{\partial R}{\partial x^i} = \left(R_i^k - \frac{1}{2} \delta_i^k R \right)_{;k} = 0. \quad (7)$$

Taking (1) into account, it is necessary to conclude that

$$T_{i;k}^k = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T_i^k}{\partial x^k} - \frac{T^{kl}}{2} \frac{\partial g_{kl}}{\partial x^i} = 0. \quad (8)$$

However, this equation, which is supposed to express the law of conservation of the energy-momentum of matter, is meaningless. First, in a gravitational field the 4-momentum of matter (including the electromagnetic field) and the 4-momentum of the gravitational field must be conserved, which is not taken into account by equation (8); second, the integral

$$P_i = \frac{1}{c} \int T_i^k \sqrt{-g} ds_k, \quad (9)$$

expressing the law of conservation of the 4-momentum only of matter, has meaning only when the condition

$$\frac{\partial \sqrt{-g} T_i^k}{\partial x^k} = 0, \quad (10)$$

is satisfied, and not condition (8).

Thus, the basic equation of relativity (1) turns out to be incorrect; it does not express a law of conservation of matter and the gravitational field.

Equation (1), as Einstein himself pointed out, was derived not from rigorous but from heuristic considerations, although its general structure does not give rise to doubts, since the metric of space is connected by this relation with the matter filling this space and creating its metric.

In order somehow to overcome the difficulties with conservation laws, which are not described by equation (1), Einstein also proposed introducing the so-called pseudotensor (t_i^k) of the energy-momentum of the gravitational field itself. By introducing this pseudotensor it was possible, in an artificial way, somehow to satisfy the integral conservation laws of 4-momentum. (It is true that a number of relativists asserted, and still assert, that general relativity is above conservation laws, stands over them, and that their fulfillment is not at all obligatory in Einstein's theory. Such argumentation can hardly be regarded as convincing.)

One method of introducing the pseudotensor into the conservation law is given in (1), where t_i^k is determined from the relations

$$\frac{16\pi G}{c^4}(-g)(T^{ik} + t^{ik}) = \frac{\partial^2}{\partial x^l \partial x^m} [(-g)(g^{ik}g^{lm} - g^{il}g^{km})], \quad (11)$$

and, identically, the equations

$$\frac{\partial}{\partial x^k} [(-g)(T^{ik} + t^{ik})] = 0. \quad (12)$$

The quantities t^{ik} are symmetric in the indices i and k , and in the general case they do not have tensor character, since t^{ik} are defined through ordinary, not covariant, derivatives. However, since t^{ik} can be expressed through the quantities Γ_{kl}^i , under linear transformations t_i^k behaves as a tensor. Sometimes t_i^k is introduced by varying the pseudoscalar of the field

$$G = g^{ik}(\Gamma_{il}^m \Gamma_{km}^l - \Gamma_{ik}^l \Gamma_{lm}^m).$$

With the identities (12), a conservation law holds for the quantities:

$$P^i = \frac{1}{c} \int (-g)(T^{ik} + t^{ik}) ds_k. \quad (13)$$

However, such a definition of the energy-momentum of the gravitational field is not unique or satisfactory. The gravitational field itself is not included in the basic equation (1) and therefore seemingly does not influence the metric of space, which is incomprehensible, since the gravitational field and gravitational waves, described by the pseudotensor t_i^k , possessing energy, themselves create an additional gravitational field, i.e. change the metric of space.

Let us also note that in the study of gravitational waves there exists, depending on the method of derivation and the choice of coordinate system ⁽²⁾ of the energy-momentum pseudotensor, an arbitrary possibility either to ascribe to the waves the transport of some definite energy, or, on the contrary, to consider that the transported energy is absent ⁽²⁾, which, of course, is absurd.

In all formal attempts to develop general relativity it was assumed that the quantity $\varkappa = 8\pi G/c^4 = \text{const}$. It is true that Dirac, in his time, expressed the interesting supposition that the gravitational constant G may not be constant, but may vary with time ^(3,4). However, his ideas were of a general descriptive, not formal, character (for Dirac $G \sim \tau^{-1}$, where τ is a certain world time).

We shall make an attempt at a certain generalization of general relativity, more precisely, a generalization of the basic gravitational equation (1), assuming that

$$\varkappa = \varkappa(g_{ik}; R; x^i).$$

Let us write the basic equation, which is easily derived by the variational method, assuming that the scalar $R^* = B^{ik}R_{ik}$, which has the dimension of energy density and is the Lagrangian of the field, is varied:

$$R_i^k - 1/2 \delta_i^k R = A_i^r (T_r^k + t_r^k) + A_i^k R_l^m \frac{\partial B_m^l}{\partial \ln g} + \frac{1}{2} (A_i^k B_r^l R_l^r - \delta_i^k R), \quad (14)$$

here A_i^r is a tensor generalizing the “gravitational constant” \varkappa ; $A_i^r B_r^k = \delta_i^k$, T_r^k is the energy-momentum tensor of matter (particles with nonzero rest mass, including “gravitons”), t_r^k is the tensor or pseudotensor of the energy-momentum of fields, including the free gravitational field (particles with zero rest mass).

Since $(R_i^k - 1/2 \delta_i^k R)_{;k} = 0$, the equation

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} A_i^r (T_r^k + t_r^k)}{\partial x^k} - \frac{A_r^l}{2} (T^{kr} + t^{kr}) \frac{\partial g_{kl}}{\partial x^i} + \psi_{i;k}^k = 0, \quad (15)$$

will hold, where

$$\psi_i^k = A_i^k R_l^m \frac{\partial B_m^l}{\partial \ln g} + \frac{1}{2} (A_i^k B_l^r R_r^l - \delta_i^k R).$$

In order that the conservation law for the 4-momentum

$$P_r = \frac{1}{c} \int \sqrt{-g} (T_r^k + t_r^k) ds_k, \quad (16)$$

be satisfied, it is necessary to assume that

$$\frac{\partial \sqrt{-g} (T_r^k + t_r^k)}{\partial x^k} = 0. \quad (17)$$

Then, starting from (15), we shall have

$$\frac{A_r^l}{2} (T^{kr} + t^{kr}) \frac{\partial g_{kl}}{\partial x^i} = (T_i^k + t_i^k) \frac{\partial A_i^r}{\partial x^k} + \psi_{i;k}^k. \quad (18)$$

Let us now represent (17) in the form:

$$\frac{\partial \sqrt{-g} T_r^k}{\partial x^k} = - \frac{\partial \sqrt{-g} t_r^k}{\partial x^k} = f_r, \quad (19)$$

where f_r is the force acting on matter from the gravitational field (and, conversely, acting on the field as the reaction of the matter to it). Suppose that

$$A_i^r f_r = \frac{A_r^l}{2} \sqrt{-g} T^{kr} \frac{\partial g_{kl}}{\partial x^i}. \quad (20)$$

Then, from (19) and (20), we shall have

$$A_i^r \frac{\partial \sqrt{-g} T_r^k}{\partial x^k} = \frac{A_r^l}{2} \sqrt{-g} T^{kr} \frac{\partial g_{kl}}{\partial x^i}. \quad (21)$$

It should be emphasized that the system of equations (14), (17), (18), (21) is complete and determines the components of the tensors T_i^k , t_i^k , A_i^k , g_{ik} .

Let us now consider a simpler case, assuming that $\varkappa \neq \text{const}$ and does not have tensor properties; \varkappa is again a scalar, but variable. Then equation (14) can be written in the form:

$$R_i^k - 1/2 \delta_i^k R = \varkappa (T_i^k + t_i^k) - \delta_i^k R \frac{\partial \ln \varkappa}{\partial \ln g}. \quad (22)$$

On contracting, we have $+R = -\varkappa T + 4R \frac{\partial \ln \varkappa}{\partial \ln g}$, where $t = t_i^i = 0$.

Covariant differentiation gives the equation

$$\frac{\partial \sqrt{-g} (T_i^k + t_i^k)}{\sqrt{-g} \partial x^k} - \frac{1}{2} (T^{kl} + t^{kl}) \frac{\partial g_{kl}}{\partial x^i} + (T_i^k + t_i^k) \frac{\partial \ln \varkappa}{\partial x^k} - \frac{1}{\varkappa} \frac{\partial \psi}{\partial x^i} = 0, \quad (23)$$

where $\psi = R \partial \ln \varkappa / \partial \ln g$.

Let

$$\frac{\partial \sqrt{-g} (T_i^k + t_i^k)}{\partial x^k} = 0; \quad (24)$$

then the 4-momentum will be conserved:

$$P_i = \frac{1}{c} \int \sqrt{-g} (T_i^k + t_i^k) ds_k. \quad (25)$$

In this case

$$\frac{1}{2} (T^{kl} + t^{kl}) \frac{\partial g_{kl}}{\partial x^i} = (T_i^k + t_i^k) \frac{\partial \ln \varkappa}{\partial x^k} - \frac{1}{\varkappa} \frac{\partial \psi}{\partial x^i}; \quad (26)$$

$$\frac{\partial \sqrt{-g} T_i^k}{\partial x^k} = \frac{\sqrt{-g}}{2} T^{kl} \frac{\partial g_{kl}}{\partial x^i} = f_i. \quad (27)$$

These relations can also be obtained from the more general ones (see above), by putting $A_i^r = \varkappa \delta_i^r$.

Specifying $\varkappa = \theta(g_{ik})f(x^i; R)$, we find that the system of equations, in particular, will determine θ , f , and the 10 components of the tensor t_i^k , which, splitting into two groups of 5 components each, will describe the field corresponding to particles with spin 2.

Thus, the formalism being developed logically shows that the fulfillment of the conservation laws of the energy-momentum of matter and field can be achieved only under the assumption that $\varkappa \neq \text{const}$.

The generalization made of Einstein's equation may have application to cosmological models. In the case of stellar masses, near these stars the proper gravitational fields are always greater than the general background and therefore, for example, the effects of the Schwarzschild problem will not undergo changes.

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Note: Figure translations are in progress. See original paper for figures.

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