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1963

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Abstract

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An Isolated Singular Point on a Surface with a Regular Metric of Negative Curvature

(Presented by Academician P. S. Aleksandrov on 29 X 1962)

1°. Formulation of the results.

We shall consider, in three-dimensional Euclidean space E_3, a surface S of negative Gaussian curvature K with an isolated singular point O. We shall call a singular point **weakly irregular** if the smoothness of the surface and the continuity of K are preserved at it, while the second quadratic form is discontinuous. Cohn-Vossen proved that for $K < 0$ a weakly irregular point can exist on a surface with a regular and even with an analytic metric ([6], p. 169). As is known (see ([4,8])), a surface of negative curvature in a neighborhood of a weakly irregular point O is a saddle of order $m \geqslant 2$, i.e., the intersection of S with the tangent plane drawn at the point O consists of $2m$ smooth arcs issuing from O.

In the present note the structure of a neighborhood of a weakly irregular point on a surface with a regular metric is investigated, and the following results are established:

Theorem 1. *If $S \in C^1$, $S \setminus O \in C^4$, the metric of S belongs to the class C^4 and has negative curvature, then the order of the saddle at the point O is equal to two.*

Theorem 2. *If the hypotheses of Theorem 1 are satisfied and, in addition, the mean curvature H is bounded on $S \setminus O$, then regularity cannot be lost at the point O. Under these conditions S is twice differentiable with the Lipschitz condition on the second derivatives.*

Let us note that if in the hypothesis of Theorem 1 the regularity of the metric is replaced by the continuity of K, then the order of the saddle may be arbitrarily large, even if H is bounded (see ([9]), § 3). Theorem 2 is an a priori estimate of the regularity of the solution of the problem of local realization in E_3 of a regular metric of negative curvature.

The proof of Theorems 1 and 2 is based on the investigation of the properties of the asymptotic net in a neighborhood of the point O. The main steps of the proofs are given below.

2°. Notation, constants, auxiliary inequalities.

Let the metric ds^2, given on some disk U of the form $x^2+y^2 \leqslant \rho^2$, be realized in E_3 on a surface S satisfying the hypotheses of Theorem 1. By a similarity transformation we shall ensure that the inequality $K \leqslant -1$ holds. Put $k = \sqrt{-K}$, $Q_1 = \ln\sqrt{k}$, $Q_2 = 1/\sqrt{k}$, and introduce the constants:

$$c_k^{(0)} = \max_U k, \quad c_k^{(1)} = \max_U |\partial Q_1/\partial s|, \quad c_k^{(2)} = \max_U \left|\sqrt{k}\,(Q_2^3 \partial^2 Q_2/\partial s^2 - 1)\right|,$$

where the derivatives are taken with respect to the arc length of geodesics, and the maximum is over all points of U and all directions at each point.

Introduce in E_3 rectangular Cartesian coordinates (x,y,z), taking the point O as the origin and as the z-axis the normal to S at this point. The surface S in a neighborhood of O is given by an equation of the form $z = f(x,y)$. The first- and second-order partial derivatives of f with respect to x and y will be denoted by p,q,r,s,t. The degree of the mapping of a neighborhood of O onto the Gaussian sphere will be denoted by N.

In asymptotic parameters (u,v) the line element of $S \setminus O$ has the form

$$ds^2 = e^2 du^2 + 2eg\cos\omega\, du\, dv + g^2 dv^2,$$

where ω is the net angle of the asymptotics, $0<\omega<\pi$. According to the sign of the geodesic torsion, the asymptotic lines on $S \setminus O$ split into two families $\mathcal{A}_1$ and $\mathcal{A}_2$. The first of them ($v = \text{const}$) will be regarded as the family with positive torsion. For a line of the j-th family ($j=1,2$) introduce the following notation: $\vec{\tau}_j$ is the unit tangent vector; Θ_j is the angle between the x-axis and the projection of $\vec{\tau}_j$ onto

the plane (x,y), defined up to π; $\varkappa_j$ is the geodesic curvature, $\varkappa_j^*$ the spatial curvature; $\partial/\partial s_j$ is differentiation with respect to arc length. The index of the asymptotic net at the point O will be called the quantity $I = \Delta\theta_j/\pi$, where $\Delta\theta_j$ is the increment of θ_j under one positive circuit about O (we note that $\Delta\theta_1 = \Delta\theta_2$). Since $S \setminus O \in C^4$ and $ds^2 \in C^4$, the formulas of surface theory in asymptotic parameters ([5]) are applicable; from them the following estimates follow:

$$\left|\frac{\partial \ln(ek)}{\partial s_2}\right| \leqslant c_k^{(1)}, \qquad \left|\frac{\partial \ln(gk)}{\partial s_1}\right| \leqslant c_k^{(1)}; \tag{1}$$

$$\left|\frac{\partial}{\partial s_2}\left[Q_2^3 (ek)^3 \varkappa_1\right]\right| \leqslant c_k^{(2)} (ek)^3, \qquad \left|\frac{\partial}{\partial s_1}\left[Q_2^3 (gk)^3 \varkappa_2\right]\right| \leqslant c_k^{(2)} (gk)^3; \tag{2}$$

$$\left|\frac{\partial \omega}{\partial s_j}\right| \leqslant c_k^{(1)} + |\varkappa_j|. \tag{3}$$

Using, in addition to ([5]), the Frenet formulas for asymptotic lines and the Beltrami-Enneper theorem ([3], pp. 83 and 126), we find:

$$\left|\frac{\partial \vec{\tau}_j}{\partial s_n}\right| \leqslant c_k^{(0)} + c_k^{(1)} \quad (j, n = 1, 2;\ j \neq n). \tag{4}$$

Next, for an asymptotic quadrilateral with area σ and lengths of opposite sides l_1 and l_3 on $S \setminus O$, the estimate ([5]) is valid (see ([10])):

$$l_3 \leqslant c_k^{(0)} \left(l_1 + c_k^{(1)} \sigma\right). \tag{5}$$

Below we shall need expressions for r, s, and t in terms of p, q, k, and θ_j:

$$r = \frac{2k(1 + p^2 + q^2) \tan\theta_1 \, \tan\theta_2}{\tan\theta_1 - \tan\theta_2}, \qquad t = -\frac{2k(1 + p^2 + q^2)}{\tan\theta_2 - \tan\theta_1}, \tag{1}$$

$$s = -\frac{k(1 + p^2 + q^2)(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}. \tag{6}$$

3°. The index of the asymptotic net.

Shil' t established ([13]) that on a regular surface of nonpositive curvature the index of the asymptotic net at an isolated zero K is related to the order of the saddle by the formula $I = 2 - m$. (Shil' t calls the quantity $m - 1$ the order of the saddle.) We shall show that in our case Shil' t' s formula is valid.

Lemma 1. *If $S \in C^1$, $S \setminus O \in C^2$, and $K < 0$ on $S \setminus O$, then $I = 2 - m \leqslant 0$.*

Proof. It is not difficult to show (see ([4])) that the area of the spherical image S (taking account of the multiplicity of covering) is bounded and that $N = 1 - m$. Therefore $m \geqslant 2$ (see ([8], pp. 51–53)). The same degree N is possessed by the mapping of the plane (x, y) onto the plane (x_1, y_1) given by the formulas

$$x_1 = q(x, y), \qquad y_1 = -p(x, y) \tag{7}$$

(an analogous mapping was considered in ([4], §48)). The mapping (7) is continuous, and for $x^2 + y^2 > 0$ belongs to the class C^1, has Jacobian $rt - s^2 < 0$, and transforms the angle θ_1 into the angle $\theta_1^*(x_1, y_1) = \theta_1(x, y)$. Therefore the Kneser-Fossen lemma is applicable to (7), which establishes that $I = N + 1$ ([6], pp. 52–62). Expressing N in terms of m, we obtain the assertion of Lemma 1.

4°. Asymptotic lines in a neighborhood of O.

In what follows we shall assume that S satisfies the conditions of Theorem 1. The question of the structure of A_j in a neighborhood of O reduces to the study of solutions of a system of differential equations on the auxiliary plane (x_2, y_2). Indeed, let R be the two-sheeted Riemann surface with branch point O covering the plane (x, y). The angle θ_j on R is continuous and is defined up to a summand that is a multiple of 2π. The projection of A_j onto $R \setminus O$ coincides with the set of trajectories of the system

$$dx/dt = 2\sqrt{x^2+y^2}\cos\theta_j,$$

$$dy/dt = 2\sqrt{x^2+y^2}\sin\theta_j.$$

The transformation $x_2 + iy_2 = \sqrt{x+iy}$ transforms this system into a system of the form (8):

$$dx_2/dt = X(x_2, y_2), \qquad dy_2/dt = Y(x_2, y_2). \tag{8}$$

Set $X(0,0) = Y(0,0) = 0$. Then X and Y are continuous, $X^2 + Y^2 = x_2^2 + y_2^2$; the uniqueness of solutions of (8) for $x_2^2 + y_2^2 > 0$ follows from the regularity of $S \setminus O$, and for $x_2 = y_2 = 0$ it is verified directly. Therefore Bendixson' s theorems are applicable to (8) (see ([2]) and ([7]), Ch. 2, § 3), from which the following follows.

Since $I \leqslant 0$, in a neighborhood V of the point O on S the curves of $\mathcal{A}_j$ both of whose ends lie on the boundary of V fill a nonempty open set. It consists of a finite number γ of connected regions, called hyperbolic. With a suitable choice of V, the boundary of each of the hyperbolic regions contains O (we shall assume that V is chosen precisely so; the neighborhood V, generally speaking, is not circular). $\gamma = \gamma(V)$ does not decrease as V decreases and satisfies the inequality

$$\gamma(V) \geqslant 2 - I. \tag{9}$$

Let G be a hyperbolic region of the family $\mathcal{A}_j$; let $\overline{G}$ be its closure; let Γ be the part of its boundary lying in V, and, for definiteness, let $j = 1$. The line Γ is the limit of a uniformly convergent sequence of lines of $\mathcal{A}_1$ lying in G, and consists of the point O and a finite or countable set of arcs of $\mathcal{A}_1$. Two of them (L_1 and L_2) have one end at O, the other on the boundary of V, while the remaining ones $(\lambda_1, \lambda_2, \ldots)$ have both ends at O. Without loss of generality one may assume that the boundary of G consists of Γ, two lines of $\mathcal{A}_2$, and one of $\mathcal{A}_1$. In $\overline{G} \setminus O$ one can introduce asymptotic coordinates (u, v). Below it is shown that $\Gamma = L_1 \cup O \cup L_2$.

5°. **Investigation of the region G.** Relying on inequalities (1), (2), (3), and (5), we successively establish properties a)–d).

a) The length of Γ and the lengths of all asymptotic segments in $\overline{G}$ are bounded by a certain constant c_1 (this follows from (5)).

b)
$$\sup_{\overline{G}}\{|\ln(ek)|,\ |\ln(gk)|\} = c_2 < \infty$$
(is derived from (1) and a)).

c)
$$\sup_{\overline{G}} |\varkappa_j| \leqslant c_\varkappa < \infty$$
(is derived from (2), a), and b)).

d) If an asymptotic curve lies in $\overline{G}$ and has an end at O, then it has a tangent half-line at O (this follows from a), c), and the equality $\varkappa_j^* = |\varkappa_j|$).

e) The number of curves λ_n does not exceed $c_3 = c_1 c_\varkappa/\pi$ (this follows from a), c), d), and the Gauss–Bonnet formula applied to $\lambda_n \cup O$).

Property e) makes it possible to decrease V so much that the boundary of each of the hyperbolic regions in V contains no arcs of the form λ_n. Then
$$\Gamma = L_1 \cup O \cup L_2.$$

Lemma 2. *If S satisfies the conditions of Theorem 1 and the neighborhood V is sufficiently small, then G contains a unique line of $\mathcal{A}_2$ having an end at O.*

Proof. Those lines of $\mathcal{A}_2$ which intersect L_1 and L_2 fill nonintersecting regions G_1 and G_2, and those which pass through points of the set $P = G \setminus (G_1 \cup G_2)$ "enter" O. Let M_n be such a sequence of lines of $\mathcal{A}_1$ that $M_n \subset G$ and $M_n \to \Gamma$ as $n \to \infty$, and let l_n be the length of the arc $M_n \cap P$. The set P is bounded in G by the lines P_1 and P_2 of $\mathcal{A}_2$ entering O (by Bendixson' s theorem). Therefore $M_n \cap P$ contracts to the point O as $n \to \infty$, and since the convergence is uniform, from c) we have:
$$\lim_{n\to\infty} l_n = 0$$
(([1]), p. 229). But then from b) we obtain that $l_n = 0$ identically, i.e. $P_1 = P_2 = P$.

Lemma 3. *Under the conditions of Lemma 2 the curve Γ is smooth.*

Proof. By property d), the curves L_1 and L_2 form at O some angle α. Apply the Gauss–Bonnet formula to G, and, taking into account (3), (5), and c), find that $|\pi - \alpha| \to 0$ as $G \to O$; hence $\alpha = \pi$.

6°. **Proof of Theorem 1.** From Lemma 3 there follows the estimate $\gamma(V) \leqslant 2$, which together with (9) and Lemma 1 gives $m = \gamma(V) = 2$.

Corollary. At the point O the first and second asymptotic directions are defined –the common tangent boundaries of two hyperbolic domains of each of the families $\mathcal{A}_i$.

We shall denote by ω_0 the angle between the asymptotic directions at O.

7°. **The case of bounded H.** Let $|H| \leqslant c_H$ in $S \setminus O$.

Lemma 4. *Under the hypotheses of Theorem 2, through the point O there passes a unique asymptotic of each family, and the angle ω is continuous at O.*

Proof. We note that

$$\lim_{A \to O} \omega(A) = \omega_0, \qquad \text{if the point } A \in \overline{G} \setminus O. \tag{10}$$

(10) follows from the Gauss–Bonnet formula applied to an asymptotic quadrilateral $D \in \overline{G}$ with vertex at O as $D \to O$, and from (3), (5), c) and Lemma 3. Since $\sin\omega = k(k^2 + H^2)^{-1/2}$, from (10) we have: $\sin\omega_0 \geqslant \lim_{A \to O} \sin\omega(A) \geqslant (c_k^{(0)} + c_H)^{-1}$, whence $\omega_0 > 0$. Consequently, the set of points separating two hyperbolic domains of one family lies (with the exception of O) inside the hyperbolic domains of the other family and, by Lemma 2, consists of a single curve, which is the desired asymptotic. The continuity of ω follows from (10).

Corollary. On S in a neighborhood of the point O one can introduce asymptotic coordinates (u, v).

Remark. On a surface of negative curvature of class C^2 there need not be uniqueness of asymptotics [12].

Lemma 5. *Under the hypotheses of Theorem 2, the functions $\theta_j(x, y)$ $(j = 1, 2)$ in a sufficiently small neighborhood of the point O satisfy the Lipschitz condition.*

For the proof it is necessary to estimate the difference of the values of θ_j at two points in the coordinates (u, v) with the aid of (4) and c), and then pass to the coordinates (x, y), using b) and the continuity of ω.

Proof of Theorem 2. We take as the x-axis the bisector of the angle between the asymptotic directions at the point O. Then $\theta_2(0,0) = -\theta_1(0,0) = \frac{1}{2}\omega_0 > 0$. Applying (6), Lemma 4 and Lemma 5, we obtain:

$$\lim_{x,y \to 0} r(x,y) = k_0 \tan\frac{\omega_0}{2}; \qquad \lim_{x,y \to 0} s(x,y) = 0;$$

$$\lim_{x,y \to 0} t(x,y) = -k_0 \cot\frac{\omega_0}{2}, \tag{11}$$

where k_0 is the value of k at the point O. By the theorem on the limit of derivatives ([11], p. 228), at the point O there exist r, s, and t, respectively equal to the limiting values (11). Consequently, $S \in C^2$, and p and q satisfy the Lipschitz condition. Applying once more (6) and Lemma 5, we find that r, s, and t satisfy the Lipschitz condition.

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Received
24 X 1962

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Note: Figure translations are in progress. See original paper for figures.