# MATHEMATICS

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## Abstract

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# ON THE UNIQUE DETERMINACY OF A PIECEWISE REGULAR SURFACE OF POSITIVE CURVATURE WITH A BOUNDARY CONDITION

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Consider a regular ovaloid $S$ of class $D_{3,p}$ $(p > 2)$ (the radius vector of the surface $\bar{r}(u, v)$ admits three generalized Sobolev derivatives summable to the power $p$), with boundary $\Gamma \in C^1_\mu$ $(0 < \mu < 1)$, containing a certain linear set $\gamma$ of nonregular points, where discontinuities of the second derivatives are allowed. Suppose, moreover, that the line $\gamma$, belonging to the class $C^1_\mu$, divides the surface $S$ into two pieces: a simply connected $S^+$ and a doubly connected $S^- \supset \Gamma$, glued along $\gamma$ in such a way that the latter is their line of contact (the angle between the corresponding strips of the curve $\gamma$ is zero). The Gaussian curvature of $S$ is positive up to the boundary $\Gamma$. In the present note we give boundary conditions, analogous to ($^1$), imposed on the normal curvature and the geodesic torsion of the contour $\Gamma$, under which the surface $S$ is uniquely determined.

1. Introduce on $S$ a single isothermally conjugate parametrization, homeomorphically mapping it onto a domain $D = D^+ + D^-$ of the plane of the complex variable $z = x + iy$, where $D^+$ is situated inside the curve $l$ $(x = x(s), y = y(s))$, into which $\gamma$ is mapped, and $D^-$ is the curvilinear annulus between $l$ and the image of $\Gamma$, the curve $L$. The curves $l$ and $L$ belong to $C^1_\mu$.

Without loss of generality one may assume that $L$ is the unit circle $x = \cos\varphi$, $y = \sin\varphi$. This can be achieved by a conformal transformation preserving the introduced parametrization.

The fundamental forms of the surface $S$ take the form:

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2; \qquad II = b_0(dx^2 + dy^2),$$

where $E, F, G, b_0$ are, in general, piecewise-continuous functions given in the domain $D$, and $b_0$ is assumed $> 0$. In what follows, a function $f(z)$ given in $D$ will be denoted by $f^+(z)$ if $z \in D^+$, and by $f^-(z)$ if $z \in D^-$. The limiting values of $f(z)$ on the contour will be denoted respectively by $f^+(t)$ and $f^-(t)$.

Suppose that under an isometric transformation of the surface $S$ the coefficients of the second form receive increments $\Delta L, \Delta M, \Delta N$, and the normal curvature and geodesic torsion of the contours receive increments $\Delta k_n$ and $\Delta \tau_g$. The former then satisfy the Gauss and Peterson-Codazzi equations:

$$(b_0 + \Delta L)(b_0 + \Delta N) - \Delta M^2 = Ka, \qquad a = EG - F^2,$$

$$\Delta L_y - \Delta M_x - \Gamma_{11}^1 \Delta L + (\Gamma_{11}^1 - \Gamma_{12}^2)\Delta M + \Gamma_{11}^2 \Delta N = 0, \tag{1}$$

$$\Delta N_x - \Delta M_y + \Gamma_{22}^1 \Delta L + (\Gamma_{22}^2 - \Gamma_{12}^1)\Delta M - \Gamma_{12}^2 \Delta N = 0.$$

To obtain a relation between the increments of the coefficients of the second form, we take into account the conditions of conjugacy on the gluing line. From the formulas for the rotation of a strip ($^2$), taking the gluing angle $\theta$ to be zero, we obtain $\Delta\theta = 0$, and, consequently, the conjugacy conditions on $l$ have the form:

$$\Delta k_n^+(t) = \Delta k_n^-(t); \qquad \Delta \tau_g^+(t) = \Delta \tau_g^-(t) \quad (t \in l). \tag{2}$$

Set $\Delta M = U,\ \Delta L = \Pi + V,\ \Delta N = \Pi - V$.
Gauss' equation gives, since $K > 0$:

$$\Pi = -b_0 + \sqrt{b_0^2 + U^2 + V^2}.$$

Representing the function $\Pi$ in the form

$$\Pi = q_1(b_0, U, V)U + q_2(b_0, U, V)V, \tag{3}$$

where

$$q_1 = \frac{U}{b_0 + \sqrt{b_0^2 + U^2 + V^2}}, \qquad q_2 = \frac{V}{b_0 + \sqrt{b_0^2 + U^2 + V^2}},$$

and introducing the notation

$$I^\pm = E^\pm \dot{x}^2 + 2F^\pm \dot{x}\dot{y} + G^\pm \dot{y}^2,$$

$$\sqrt{a^\pm}\, I^\pm \alpha^\pm = E^\pm \dot{x}^2 - G^\pm \dot{y}^2 + \left[-F^\pm(\dot{x}^2 - \dot{y}^2) + (E^\pm - G^\pm)\dot{x}\dot{y}\right] q_1^\pm,$$

$$\sqrt{a^{\pm}}\,I^{\pm}\beta^{\pm}=-F^{\pm}(\dot{x}^{2}+\dot{y}^{2})-(E^{\pm}+G^{\pm})\dot{x}\dot{y}+\left[-F^{\pm}(\dot{x}^{2}-\dot{y}^{2})+(E^{\pm}-G^{\pm})\dot{x}\dot{y}\right]q_{2}^{\pm},$$

$$I^{\pm}\gamma^{\pm}=2\dot{x}\dot{y}+(\dot{x}^{2}+\dot{y}^{2})q_{1}^{\pm},$$

$$I^{\pm}\delta^{\pm}=(\dot{x}^{2}-\dot{y}^{2})+(\dot{x}^{2}+\dot{y}^{2})q_{2}^{\pm},\tag{4}$$

we write condition (2) in the following form:

$$\alpha^{+}U^{+}+\beta^{+}V^{+}=\alpha^{-}U^{-}+\beta^{-}V^{-},$$

$$\gamma^{+}U^{+}+\delta^{+}V^{+}=\gamma^{-}U^{-}+\delta^{-}V^{-}.\qquad (t\in l)\tag{5}$$

On the contour $L$, analogously to (1), we obtain the relation

$$\left(\mu\sqrt{\frac{a^{-}}{e^{-}}}\alpha^{-}+\lambda\gamma^{-}\right)U^{-}+\left(\mu\sqrt{\frac{a^{-}}{e^{-}}}\beta^{-}+\lambda\delta^{-}\right)V^{-}=\lambda\Delta k_{n}^{-}+\mu\frac{\sqrt{a^{-}}}{e^{-}}\Delta\tau_{g}^{-}=\sigma.\tag{6}$$

Here $\lambda$ and $\mu$ are certain prescribed functions of points of the contour $L$, belonging to the class $C_{\mu}^{1}(L)$; the coefficients $\alpha,\beta,\gamma,\delta$ have the same form as in (4), but refer to the contour $L$.

We shall now consider the boundary-value problem (1), (5), (6), regarding $U,V$ as the unknown functions, and $\sigma$ as prescribed on $L$.

**2.** Following (3) and putting $w=U+iV$, $\partial_{\bar{z}}\equiv\frac{1}{2}(\partial_{x}+i\partial_{y})$, $\partial_{z}\equiv\frac{1}{2}(\partial_{x}-i\partial_{y})$, we write our boundary-value problem in complex form

$$\partial_{\bar{z}}w+A(z)w+B(z)\overline{w}=C(z)\Pi+i\partial_{z}\Pi;\tag{7}$$

$$w^{+}(t)=G_{1}(t,w^{+},w^{-})w^{-}(t)+G_{2}(t,w^{+},w^{-})\overline{w^{-}(t)}\qquad (t\in l);\tag{8}$$

$$\operatorname{Re}\left[\mu\frac{\sqrt{a^{-}}}{e^{-}}\alpha^{-}+\lambda\gamma^{-}-i\left(\mu\frac{\sqrt{a^{-}}}{e^{-}}\beta^{-}+\lambda\delta^{-}\right)\right]w^{-}(t)=\sigma(t)\qquad (t\in L).\tag{9}$$

Here $A,B,C$ are known functions of class $L_{p,2}$, expressible in terms of the Christoffel symbols,

$$2(\alpha^+\delta^+ - \beta^+\gamma^+)G_1 = \alpha^-\delta^+ - \beta^+\gamma^- + \alpha^+\delta^- - \beta^-\gamma^+$$

$$+i(\alpha^+\gamma^- - \alpha^-\gamma^+ + \beta^+\delta^- - \beta^-\delta^+), \tag{10}$$

$$2(\alpha^+\delta^+ - \beta^+\gamma^+)G_2 = \alpha^-\delta^+ - \beta^+\gamma^- - \alpha^+\delta^- + \beta^-\gamma^+$$

$$+i(\alpha^+\gamma^- - \alpha^-\gamma^+ - \beta^+\delta^- + \beta^-\delta^+).$$

**Lemma 1.** The coefficients $G_1$ and $G_2$ of the boundary condition (8), uniformly in $w^+(t), w^-(t) \subset C_\mu(l)$, satisfy the conditions:

$$1. \quad |G_1(t, w^+, w^-)| > |G_2(t, w^+, w^-)|. \tag{11}$$

$$2. \quad \operatorname{Ind} G_1(t, w^+, w^-) = 0 \qquad (t \in l). \tag{12}$$

The first assertion follows from the relation

$$|G_1|^2 - |G_2|^2 = \frac{\alpha^-\delta^-}{\alpha^+\delta^+} - \frac{\beta^-\gamma^-}{\beta^+\gamma^+}$$

and the obvious inequality

$$[2\dot{x}\dot{y}V - (\dot{x}^2 - \dot{y}^2)U]^2 > 0.$$

The second—from the fact that

$$\operatorname{Re} G_1(t, w^+, w^-) > 0 \tag{13}$$

uniformly in $w^+, w^-$.

**Lemma 2.** The index $\varkappa^-$ of the boundary condition (9) is connected with $\operatorname{Ind}(\mu + i\lambda)$ by the relation

$$\varkappa^- = -2 + \operatorname{Ind}(\mu + i\lambda) \tag{14}$$

uniformly in $w^+, w^-$.

**3.** Using the basic integral representation (4) of solutions of system (1), we reduce problem (7), (8), (9) to a boundary-value problem for a piecewise analytic function. We have

$$w^{\pm}(z) = f^{\pm}(z)e^{\omega_{\pm}(z)+\omega_{\pm}(z)},$$

where $f^{\pm}(z)$ is a function analytic in $D^{\pm}$, $\omega(z) \in C_{\alpha}(D)$.

Such a representation is possible, since $e^{\omega^{-}}$ is analytic in $D^{+}$, while $e^{\omega^{+}}$ is analytic in $D^{-}$. Our problem has been reduced to the following:

$$f^{+}(t) = G_1(t)f^{-}(t) + g_2(t)\overline{f^{-}(t)} \quad (t \in l); \tag{15}$$

$$\mathrm{Re}[\overline{p(t)}f(t)] = \sigma(t) \quad (t \in L), \tag{16}$$

where $g_2(t) = G_2(t)e^{\overline{\omega}-\omega}$. In this case

$$|G_1| > |g_2|, \qquad \mathrm{Ind}\, p(t) = \varkappa^{-}. \tag{17}$$

**Lemma 3.** Let

$$\sup_{t\in l}\left|\frac{g_2(t)}{G_1(t)}\right| < \frac{2}{1+M_p},$$

where $M_p$ is the norm of the singular operator $Sv = \dfrac{1}{\pi i}\displaystyle\int \frac{v(\tau)}{\tau - t}\,d\tau$ (5) in $L_p$. Then, for $\varkappa^{-} \geqslant 0$, problem (15), (16) has a solution depending on $2\varkappa^{-}+1$ real constants. For $\varkappa^{-} < 0$, the problem may have a solution if $2|\varkappa^{-}|+1$ additional conditions are satisfied, and the homogeneous problem ($\sigma(t) = 0$) has only the zero solution. The latter is true under the weaker assumption $|G_1| > |g_2|$.

**Proof.** Let $\varkappa^{-} \geqslant 0$ and let $F(z)$ be a solution of problem (15), obtained by the method (5) with the use of the canonical function $X(z)$, satisfying the conditions:

$$X^{+}(t) = G_1(t)X^{-}(t) \quad (t \in l); \qquad \mathrm{Im}\, X^{-}(t) = q\,\Pi \quad (t \in L).$$

Then $f(z)$ is constructed with the help of $F(z)$ and of a certain function analytic in $D$ with prescribed real part on $L$, determined up to an imaginary constant. For $\varkappa^{-} < 0$, in order for the solution to be analytic, $\sigma(t)$ must satisfy $2|\varkappa^{-}|+1$ integral conditions. The last assertion of the lemma is proved by contradiction: using the canonical

function

$$X^{+}(t) = \left[1 + \frac{g_2(t)}{G_1(t)}\,\frac{\overline{f^{-}(t)}}{f^{-}(t)}\right] X^{-}(t) \quad (t \in l); \qquad \mathrm{Im}\, X^{-}(t) = 0 \quad (t \in L),$$

and also, using (12) and (17), we reduce (15), (16), for $\sigma = 0$, to $f(z) = \psi(z)X(z)$:

$$\psi^+(t) = G_1(t)\psi^-(t) \quad (t \in l); \qquad \operatorname{Re}\psi^-(t) = 0 \quad (t \in L)$$

and thence to $f(z) \equiv 0$.

From Lemmas 2 and 3 it follows:

**Theorem.** *Under the condition*

$$\sigma = 0, \qquad \operatorname{Ind}(\mu + i\lambda) < 2 \tag{18}$$

*the surface $S = S^+ + S^-$ is uniquely determined.*

4. The contour $\gamma$ may consist of open arcs $\gamma_k$: $\gamma = \sum \gamma_k$. In this case condition (18) is modified as follows:

$$\operatorname{Ind}(\mu + i\lambda) < 2 + \sum \chi_k,$$

where $\chi_k$ is the index of the auxiliary Riemann problem for the corresponding arc $l_k$ with endpoints $a_k$ and $b_k$. As in the case of infinitesimal bendings, in consequence of (13), it behaves in the following way:

$$\begin{array}{ll} \text{if } \operatorname{sign}\operatorname{Im} G_1(a_k) = -\operatorname{sign}\operatorname{Im} G_1(b_k), & \chi_k = \operatorname{sign}\operatorname{Im} G_1(a_k); \\ \text{if } \operatorname{sign}\operatorname{Im} G_1(a_k) = \operatorname{sign}\operatorname{Im} G_1(b_k), & \chi_k = 0. \end{array}$$

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*Note: Figure translations are in progress. See original paper for figures.*