Hydromechanics

L. V. CHERKESOV

1963

SovietRxiv

Abstract

Full Text

Hydromechanics

L. V. CHERKESOV

SHIP WAVES IN A VISCOUS FLUID

(Presented by Academician I. I. Artobolevskii, July 9, 1963)

In the present note we consider the spatial problem of waves on the surface of a viscous incompressible fluid, arising under the rectilinear motion along the free surface, with velocity c, of a certain system of normal stresses. The general plane problem of waves on the surface of a viscous fluid was considered in the work of L. N. Sretenskii (^{1}).

§ 1. Let a system of normal stresses of the form $p_{nn} = -p_0(x, y)$ be applied to the free surface of a viscous incompressible fluid occupying the half-space $z < 0$ and moving with velocity c in the positive direction of the x-axis. Assuming the velocities of the perturbed motion u, v, w to be small, we obtain the hydrodynamic equations in the form

$$c\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u, \qquad c\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta v,$$

$$c\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\Delta w, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

These equations are satisfied by solutions of the form

$$u = -\varphi_x - \psi_y, \qquad v = -\varphi_y + \psi_x + f_z,$$

$$w = -\varphi_z - f_y, \qquad p = \rho(c\varphi_x - gz),$$

where the functions f, φ, ψ satisfy the equations

$$\Delta\varphi = 0, \qquad c\frac{\partial f}{\partial x} = \nu\Delta f, \qquad c\frac{\partial \varphi}{\partial x} = \nu\Delta\psi. \tag{1}$$

On the free surface $z = \zeta(x, y)$ the dynamical conditions must be satisfied:

$$p_{nn} = -p_0(x, y), \qquad p_{n\tau_1} = p_{n\tau_2} = 0,$$

where τ_1 and τ_2 are two orthogonal directions. Satisfying these conditions up to small quantities of second order, and the kinematic condition

$$w = c\zeta_x \quad \text{for } z = 0,$$

we obtain three boundary conditions for the functions f, φ, ψ at $z = 0$:

$$c^2\varphi_{xx} + g(\varphi_z + f_y) + 2c\nu(\varphi_{zzx} + f_{yzx}) = \frac{c}{\rho}p_{0x},$$

$$2\varphi_{zx} + \psi_{yz} + f_{yx} = 0, \tag{2}$$

$$-2\varphi_{yz} + \psi_{xz} + f_{zz} - f_{yy} = 0.$$

In this case the form of the free surface is given by the formula

$$\zeta = \frac{1}{g}\left[c\varphi_x + 2\nu(\varphi_{zz} + f_{yz})\right]_{z=0} - \frac{1}{\rho g}p_0.$$

Applying the Fourier transform with respect to the variables x and y to equations (1) and the boundary conditions (2), and requiring that f, φ, and ψ tend to zero as $z \to -\infty$, we obtain the following expression for the elevation of the liquid:

$$\zeta = \frac{1}{2\pi\rho g}\lim_{z\to -0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\alpha(m,n,z)}{\beta(m,n)}e^{i(mx+ny)}\,dm\,dn - \frac{1}{\rho g}p_0(x,y), \tag{3}$$

$$\alpha(m,n,z) = \Pi(m,n)\left\{[cim + 2\nu(m^2+n^2)]^2\exp\left(z\sqrt{m^2+n^2}\right) + 8\nu^{3/2}b^{1/2}(m^2+n^2)^{3/2}\exp\left(z\sqrt{b\nu^{-1}}\right)\right\},$$

$$\beta(m,n) = c^2m^2 - g\sqrt{m^2+n^2} - 4c\nu im(m^2+n^2) + 2\nu^{3/2}b^{1/2}m^2\sqrt{m^2+n^2} - 4\nu^2(m^2+n^2)^2,$$

$$b = cim + \nu(m^2+n^2),$$

$\Pi(m,n)$ is the Fourier transform of the function $p_0(x,y)$.

§ 2. Formula (3) gives an exact solution of the posed problem for an arbitrary form of the disturbing normal stresses. We shall analyze this formula, assuming that $p_0(x,y)$ differs from zero only in a small neighborhood of the origin and has there a very large value, so that the total magnitude of the normal stresses at the point $(0,0)$ is equal to P. Then

$$\Pi(m,n) = \frac{1}{2\pi} P.$$

Introducing the notation

$$x = R\cos\gamma, \qquad y = R\sin\gamma, \qquad \chi = gc^{-2}, \qquad \varepsilon = \nu g c^{-3}$$

and assuming the parameter ε to be small, we obtain expression (3), accurate through terms of first order in ε, in the form

$$\zeta = \frac{\chi^2 P}{4\pi^2 \rho g} \lim_{z\to+0} \int_0^{2\pi} I(\theta)\, d\theta; \tag{4}$$

$$I(\theta) = \int_0^{\infty} \frac{\Delta_1(r,\theta)}{\Delta_2(r,\theta)} \exp\{i\chi r R\cos(\theta-\gamma) + r\chi z\}\, dr; \tag{5}$$

$$\Delta_1 = r^2\cos\theta[\cos\theta - 4\varepsilon r i], \qquad \Delta_2 = r\cos^2\theta - 1 - 4\varepsilon i r^2\cos\theta.$$

The original path of integration with respect to θ in formula (4) is replaced by the contour L, which is the segment of the real axis from the point $\theta = 0$ to $\theta = 2\pi$, bypassing the points $\theta = \pi/2, \theta = 3\pi/2$ along small semicircles of radius δ, on which $\mathrm{Re}[i\cos(\theta-\gamma)] < 0$. Such a replacement is possible, since the integrand has no singularities on the path of integration, and it is needed for convenient investigation of the roots of the equation $\Delta_2 = 0$. We calculate the integral (5) by means of residues, retaining in the pole expressions terms of order ε through first order inclusive. After this, estimating the integral (4) for large values of χR, we find the following final expression for the elevation of the liquid:

1) $0 \le |\gamma| < 19^\circ 28'$:

$$\zeta = -\frac{1}{\sqrt{\chi R}} \sum_{k=1,2} f(u_k) \left\{ \sin\left[\chi R F(u_k) + (-1)^k \frac{\pi}{4}\right] + 14\varepsilon(1+u_k^2)^{3/2} \cos\left[\chi R F(u_k) + (-1)^k \frac{\pi}{4}\right] \right\}; \tag{6}$$

$$f(u) = -\frac{\sqrt{2}\chi^2 P (1+u^2)^{7/4}}{\sqrt{\pi\rho g}\sqrt{|\cos\gamma + u(3+2u^2)\sin\gamma|}} \exp[-4\varepsilon\chi R(1+u^2)^{3/2} F(u)];$$

$$F(u) = \sqrt{1+u^2}(\cos\gamma + u\sin\gamma), \qquad u_k = \frac{-\cos\gamma + (-1)^k\sqrt{9\cos^2\gamma - 8}}{4\sin\gamma}.$$

2) $|\gamma| = 19°28'$:

$$\zeta = -\frac{3\sqrt[6]{9}\,\Gamma(1/3)\,\varkappa^2 P}{\sqrt[3]{\varkappa h}\,2\sqrt{2\pi\rho g}}\left[\sin\frac{\sqrt{3}}{2}\varkappa R + 21\sqrt{1.5\varepsilon}\cos\frac{\sqrt{3}}{2}\varkappa R\right]\exp\left(-\frac{9}{\sqrt{2}}\varkappa R\varepsilon\right).$$

3) $19°28' < |\gamma| \le \pi$:

$$\zeta = O\left[\frac{1}{\varkappa R}\exp\left(-\frac{4\varepsilon\delta}{\sin^3\gamma}\varkappa h\right)\right].$$

From the formulas obtained it is clear that, with accuracy up to and including terms of the first degree in ε, the principal disturbances of the free surface are concentrated in the region $-19°28' \le \gamma \le 19°28'$ and constitute the sum of two waves of type (6); here the first term of this sum represents longitudinal waves, and the second—transverse waves. Thus the region of the principal disturbances of the free surface, with the indicated degree of accuracy, coincides with Kelvin's wave region (^{2}), obtained in solving the analogous problem for an ideal fluid. The essential difference from the case of an ideal fluid appears in the amplitudes of the waves, since in the case of a viscous fluid the wave amplitudes contain a factor that decays exponentially with increasing distance R and viscosity coefficient ν.

Institute of Mathematics and Computer Technology
Academy of Sciences of the BSSR

Received
3 VII 1963

REFERENCES

[1] L. N. Sretenskii, *Tr. TsAGI*, No. 541 (1941). [2] L. N. Sretenskii, *Theory of Wave Motions of a Fluid*, Moscow, 1936.

Note: Figure translations are in progress. See original paper for figures.