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Abstract

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MATHEMATICS

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ON THE SOLUTION OF ILL-POSED PROBLEMS AND THE REGULARIZATION METHOD

1. Inverse problems of mathematical physics often lead to ill-posed problems. A typical example is the Fredholm equation of the first kind

$$A[x, z(s)] = \int_a^b K(x,s)z(s)\,ds = u(x), \qquad c \le x \le d. \tag{1}$$

This equation has a solution not for every function $u(x)$. It is obvious that if $K(x,s)$ has a certain degree of smoothness with respect to x, then there is no function $z(s) \in L_2$ satisfying equation (1) if $u(x)$ has a lower degree of smoothness. We shall assume uniqueness of the solution of equation (1), i.e., we shall suppose that if for some function $\bar{u}(x)$ equation (1) has a solution $\bar{z}(s)$, then it has only one.

Let the function $\bar{u}(x)$ be such that equation (1) has a solution $\bar{z}(s)$. The purpose of the present article is to set forth an algorithm for constructing a uniform approximation to the function $\bar{z}(s)$. In what follows we shall assume that $\bar{z} \in \bar{C}_1$, where $\bar{C}_1$ is the class of continuous piecewise-smooth functions.

Denote by U the class of functions $u(x) = A[x, z(s)]$, $z(s) \in \bar{C}_1$. As the norm of deviation in $\bar{C}_1$ we shall take

$$\|z\| = \max|z(s)| \qquad (a \le s \le b)$$

and as the norm of deviation in U,

$$\|u(x)\| = \left[\int_c^d u^2(x)\,dx\right]^{1/2}.$$

If the kernel $K(x,s)$ is continuous, then the mapping $\bar{C}_1 \to U$ is continuous. It should be borne in mind that the inverse problem—the problem of finding

$z(s)$ from a given function $u(x)$—is ill-posed. Indeed, to the functions $z_1(s)$ and $z_2(s) = z_1(s) + p\cos\omega s$, $z_1(s), z_2(s) \in \bar{C}_1$, where p is any fixed number (however large), there will correspond functions $u_1(x)$ and $u_2(x)$, whose norm of deviation $\|u_1(x) - u_2(x)\|$ is arbitrarily small if ω is sufficiently large. However, if the class of admissible solutions is a compact class $\bar{Z}$, then the inverse mapping $\bar{U} \to \bar{Z}$ will be stable ([1]). In other words, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon, \bar{Z})$ such that from $\|u_1 - u_2\| < \delta(\varepsilon, \bar{Z})$ it follows that $\|z_1 - z_2\| < \varepsilon$, if $u_1, u_2 \in \bar{U} = \{u(x) = A[x, z(s)],\ z \in \bar{Z}\}$, where $\bar{Z}$ is a compact class of functions.

The construction of an algorithm for obtaining an approximate solution that uniformly approximates $\bar{z}(s)$ is based on the following regularization principle: a family of functions $z^\alpha(s)$, depending on a parameter α, will be called a **regularized family of approximate solutions** if: 1) $u_\alpha(x) = A[x, z^\alpha(s)] \to \bar{u}(x)$ as $\alpha \to 0$; 2) the functions $z^\alpha(s)$, for any α, belong to a compact class of functions $\bar{Z}$ containing $\bar{z}(s)$. The regularized family of approximate solutions converges uniformly to $\bar{z}(s)$ as $\alpha \to 0$.

2. Let a function $\bar{u}(x)$ be given. Consider the functional

$$M^\alpha[z(s), \bar{u}(x)] = N[z(s), \bar{u}(x)] + \alpha\Omega[z(s)], \tag{2}$$

where the functional N represents the quadratic deviation of $\bar{u}(x)$ from $A[x, z(s)]$

$$N[z(s), \bar{u}(x)] = \int_c^d [A[x, z(s)] - \bar{u}(x)]^2\, dx,$$

$$\Omega[z(s)] = \int_a^b [k(s)z'(s)^2 + p(s)z^2(s)]\, ds \quad (k(s) > 0,\ p(s) > 0).$$

We shall call $\Omega[z]$ a **regularizing** functional and M^α a **smoothing** functional.

Theorem 1. For any function $\bar{u}(x) \in L_2$ there exists a unique continuous, differentiable function $z^\alpha(s)$ realizing the minimum of the smoothing functional $M^\alpha[z(s), \bar{u}(x)]$.

The function $z^\alpha(s)$ is determined by the Euler equation for the functional $M^\alpha[z, \bar{u}]$:

$$L^\alpha[z] = \alpha\left\{\frac{d}{ds}\left[k\frac{dz}{ds}\right] - pz\right\} - \left\{\int_a^b \bar{K}(s,\xi)z(\xi)\, d\xi - \bar{b}(s)\right\} = 0, \quad z'(a) = z'(b) = 0, \tag{3}$$

where

$$\bar{K}(s,\xi) = \int_c^d K(\xi,s)\,K(\xi,\xi)\,d\xi, \qquad \bar{b}(s) = \int_c^d K(\xi,s)\bar{u}(\xi)\,d\xi.$$

With the aid of the Green' s function for the boundary-value problem

$$L^\omega[z] = \frac{d}{ds}\left[k(s)\frac{dz}{ds}\right] - p(s)z(s) = f(s), \quad z'(a) = z'(b) = 0, \tag{4}$$

defined by the Euler operator for the regularizing functional, equation (3) can be transformed into a Fredholm equation of the second kind, for which, when $\alpha > 0$, the homogeneous equation has only the trivial solution; hence the existence of $z^\alpha(s)$ follows.

Theorem 2. If $\bar{z}(s) \in \bar{C_1}$, $u(x) = A[x, \bar{z}(s)]$, then for any $\varepsilon > 0$ there exists such an $\alpha(\varepsilon, \bar{z})$ that

$$|z^\alpha(s) - \bar{z}(s)| < \varepsilon$$

for all $\alpha < \alpha_0(\varepsilon, \bar{z})$.

Indeed,

$$M^\alpha[z^\alpha(s); \bar{u}(x)] \leq N[\bar{z}, \bar{u}] + \alpha\Omega[\bar{z}] = \alpha C^2 \quad (C^2 = \Omega[\bar{z}]),$$

whence it follows that $z^\alpha(s)$ satisfies the inequality

$$1) \quad \Omega[z(s)] \leq C^2,$$

which determines a compact class of functions $\bar{Z}$, and also

$$2) \quad \|u^\alpha(x) - \bar{u}(x)\| \leq \alpha C \to 0 \quad \text{as } \alpha \to 0.$$

Hence theorem 2 follows.

Theorem 3. If $\bar{z} \in \bar{C_1}$, then for any $\varepsilon > 0$ and any auxiliary numbers $0 < \gamma_1 < \gamma_2$ there exists such a $\delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$ that if: 1) the norm of the deviation of the function $u_\delta(x)$ from the function $\bar{u}(x)$ is less than δ

$$\|u_\delta(x) - \bar{u}(x)\| < \delta;$$

2) $\bar{\alpha} = \bar{\alpha}(\delta)$ satisfies the conditions

$$\gamma_1 \leqslant \delta^2/\alpha \leqslant \gamma_2 \quad (\text{or } \delta^2/\gamma_2 \leqslant \alpha \leqslant \delta^2/\gamma_1),$$

then $\tilde{z}_\delta^{\bar{\alpha}}(s)$, realizing the minimum of the smoothing functional $M^{\bar{\alpha}}[z, \tilde{u}_\delta(x)]$, belongs to the ε-neighborhood of the function $\bar{z}(s)$,

$$\left|\tilde{z}_\delta^{\bar{\alpha}}(s) - \bar{z}(s)\right| < \varepsilon$$

for $\delta \leqslant \delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$.

It is not difficult to verify by examples that the function $\tilde{z}_\delta^{\alpha}(s)$ corresponding to a fixed function $\tilde{u}_\delta(s)$, for small δ, as $\alpha \to 0$, may leave the ε-neighborhood of $\bar{z}(s)$.

3. Let us pass to approximate methods for solving equation (1). Consider the method of finite differences. Take a mesh on (a, b): $s_j = jh - 0.5h$ $(j = 1, \dots, n)$, and on (c, d): $x_i = ih_1 - 0.5h_1$ $(i = 1, \dots, m)$, where $h = \frac{1}{n}(a - b)$ and $h_1 = \frac{1}{m}(c - d)$. Denote $z_j = z(s_j)$, and let

$$\sum_{j=1}^{n} K_{ij} z_{jh} = \int_a^b K(x_i, s) z(s)\, ds + O(h^\gamma)$$

be some quadrature formula of order γ.

Consider the difference smoothing functional

$$\widehat{M}_h^\alpha[\hat{z}, \hat{u}] = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} K_{ij} \hat{z}_{jh} - \hat{u}_i \right\}^2 h_1 + \alpha \sum_{j=1}^{n} \left\{ k_j (\hat{z}_{j+1} - \hat{z}_j)^2 \frac{1}{h} + p_j \hat{z}_j^2 h \right\},$$

where $\hat{u} = \{\hat{u}_i\}$ is a given mesh function on $\{x_i\}$, $\hat{z} = \{\hat{z}_j\}$ is a mesh function on $\{s_j\}$, and $k_j > 0$, $p_j > 0$.

Analogously to the preceding, the following holds.

Theorem 1′. For any mesh function $\hat{u}$ and $\alpha > 0$ there exists a mesh function $\hat{z}^\alpha$ realizing the minimum of the smoothing functional $\widehat{M}_h^\alpha[\hat{z}, \hat{u}]$.

The mesh function $\hat{z}^\alpha$ is determined from the system of equations

$$\widehat{L}^\alpha[\hat{z}] = \alpha \left\{ \frac{1}{h^2} [k_j(\hat{z}_{j+1} - \hat{z}_j) - k_{j-1}(\hat{z}_j - \hat{z}_{j-1})] - p_j \hat{z}_j \right\} - \left\{ \sum_{l=1}^{n} \overline{K}_{jl} \hat{z}_{lh} - \hat{b}_j \right\} = 0, \tag{3'}$$

$$\hat{z}_0 = \hat{z}_1, \qquad \hat{z}_{n+1} = \hat{z}_n,$$

where

$$\overline{K}_{jl} = \sum_{i=1}^{m} K_{ij} K_{il} h_1, \qquad \hat{b}_j = \sum_{i=1}^{m} K_{ij} \hat{u}_i h_1,$$

and k_j and p_j are determined through $k(x)$, $p(x)$ by means of some homogeneous difference scheme converging to problem (4) (see ([2])). In particular, for example, $k_j = k(s_j + 0.5h)$, $p_j = p(s_j)$.

Theorem $2'$**.** If $z(s) \in \overline{C}_1$, then for any $\varepsilon > 0$ and any auxiliary numbers $0 < \gamma_1 \leqslant \gamma_2$ there exist such $\delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$ and $h_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$ that if: 1) the norm of the deviation of the function $\tilde{u}_\delta(x)$ from $\bar{u}(x)$ is less than δ:

$$\|\tilde{u}_\delta - \bar{u}\| < \delta;$$

2) $\bar{\alpha} = \bar{\alpha}(\delta)$ satisfies the conditions

$$\gamma_1 \leqslant \delta^2/\alpha \leqslant \gamma_2 \quad (\text{or } \delta^2/\gamma_2 \leqslant \alpha \leqslant \delta^2/\gamma_1),$$

then $\bar{z}_\delta^\alpha(s)$, which realizes the minimum of the difference smoothing functional $\widehat{M}_h^\alpha[\hat{z}, \hat{u}_\delta]$, belongs to the ε-neighborhood of the function $\bar{z}(s)$ for $\delta \leqslant \delta_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$, $h < h_0(\varepsilon, \gamma_1, \gamma_2, \bar{z})$.

Equation ($3'$) is an algorithm for solving equation (1), giving very effective results with the aid of electronic digital computers.

The construction of the functions $z^\alpha(s)$ may also be carried out using expansions in series with respect to orthogonal systems.

The method set forth above is applicable to equations of the type

$$A[x, z(s)] = u(x), \tag{1'}$$

where $A[x, z(s)]$ is a bounded operator. If we denote

$$\alpha(x, s) = A[x, \eta_s(\xi)], \qquad \eta_s(\xi) = \begin{cases} 1, & \xi \leqslant s, \\ 0, & \xi > s, \end{cases}$$

then equation (3) can be represented in the form

$$\alpha \left\{ k(s) z'(s) + \int_0^s p(\xi) z(\xi)\, d\xi \right\} - \left\{ \int_c^d A[x, z(\xi)] \alpha(x, s)\, dx - \int_c^d \alpha(x, s) \bar{u}(x)\, dx \right\} = 0,$$

$$z'(a) = z'(b) = 0.$$

The regularizing functional $\Omega[z]$ may be chosen as a quadratic functional (not necessarily differential) so that the condition $\Omega(z) \leqslant C$ determines a compact set and so that the Euler operator for $\Omega(z)$ has a completely continuous inverse operator. This also applies to the case where the domain of definition of $z(s)$ is a domain D of n dimensions (see the Sobolev–Kondrashev theorem[3]).

Smoothing functionals provide a convenient apparatus for solving equations of the second kind at an isolated point of the spectrum, as well as for solving nonlinear problems.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.