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Abstract

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ON REDUCED FREE MULTIOPERATOR GROUPS

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In the author' s note (̂1) the question of orderability of multioperator groups (̂2) was considered, and it was established that arbitrary free Ω-groups, free sums of ordered Ω-groups, and free Ω-groups with commutative addition (or free ΩA-groups (̂3)) are orderable. In the present paper the question of orderability of certain reduced free distributive Ω-groups is studied. In the second part of the article two results on orderability and finite approximability of free modules are formulated, as well as similar results on triangular extensions and free groups.

No. 1. **An orderability criterion for a distributive Ω-group.** An Ω-group G is called **ordered** if its additive group is ordered in such a way that, for every $\omega \in \Omega$ and every ordered system of $n = n(\omega)$ nonnegative elements $a_1, \dots, a_n$ of the group G, the element $a_1 \dots a_n \omega \geq 0$.

By $\Gamma(X, \Omega)$ we shall agree to denote the **absolutely free** universal algebra with system of free generators X and system of basic operations Ω.

Let a distributive Ω-group G be given. Adjoin to Ω all inner automorphisms of the additive group G. Denote the resulting set by Ω'. Form the absolutely free algebra $\Gamma(X, \Omega')$ over the finite set $X = \{x_1, \dots, x_n\}$. If $a_1, \dots, a_n$ are arbitrary nonzero elements of the group G, then by $T(a_1, \dots, a_n)$ we shall agree to denote the subsemigroup of the additive group G generated by all nonzero values of the words of the algebra $\Gamma(X, \Omega')$ under $x_1 = a_1, \dots, x_n = a_n$.

Theorem 1. *A distributive Ω-group G is orderable if and only if, for every finite collection of nonzero elements $a_1, \dots, a_n$ of G, one can choose plus or minus signs $\varepsilon_1, \dots, \varepsilon_n$ so that*

$$0 \notin T(\varepsilon_1 a_1, \dots, \varepsilon_{na} n).$$

These conditions for ordinary groups coincide with the Lorentz-Ohnishi conditions (([4,]5), see also Łoś (̂6)), and for rings—with the Jonsson-Podderyugin conditions ([7,]8). A consequence of Theorem 1 is also an orderability criterion for one class of operator groups indicated by V. D. Podderyugin (̂9).

No. 2. **Free distributive ΩA-groups.** Let X be a nonempty ordered set of symbols and let Ω be an ordered system of operations. We define the notion of the **height** of a word γ in the absolutely free algebra $\Gamma(X, \Omega)$ inductively: 1) the words of height 1 shall be the elements of the set X and only these; 2) if words of height $h \leq k$, where $k \geq 1$, have already been defined, then words of height $k+1$ will be called words of the form $\gamma_1 \dots \gamma_n \omega$, where $\omega \in \Omega$, $n = n(\omega)$, and $\gamma_1, \dots, \gamma_n$ are words of height not exceeding k, with at least one of them having height equal to k. The height of a word γ will be denoted by the symbol $h(\gamma)$. We define in the following way a linear ordering of the set of words of $\Gamma(X, \Omega)$, which we shall call the **lexicographic** ordering of the algebra $\Gamma(X, \Omega)$, determined by the given relations of linear order in the sets X and Ω.

Let Γ_k be the set of all words from $\Gamma(X, \Omega)$ whose height does not exceed k ($k = 1, 2, \dots$). By assumption the set $\Gamma_1 = X$ is ordered. Suppose that a linear order relation has already been defined on the set Γ_k, where $k \geqslant 1$.

We order the set Γ_{k+1} in the following way: if $\alpha, \beta \in \Gamma_{k+1}$, $\alpha \neq \beta$, then we put $\alpha < \beta$ if one of the following conditions is satisfied: 1) $\alpha \in \Gamma_k$, $\beta \in \Gamma_k$, and α precedes β in the set Γ_k; 2) $h(\beta) = k+1$ and $h(\alpha) \leqslant k$; 3) $\alpha = \alpha_1 \cdots \alpha_m \omega_1$, $\beta = \beta_1 \cdots \beta_n \omega_2$ are words of height $k+1$, and the multioperator ω_1 precedes the multioperator ω_2 in the set Ω, while for $\omega_1 = \omega_2$ there exists a number s such that $\alpha_1 = \beta_1, \dots, \alpha_{s-1} = \beta_{s-1}$, but the word α_s precedes the word β_s in the set Γ_k. Since the ordering in Γ_{k+1} extends the ordering in Γ_k, the set $\Gamma(X, \Omega) = \bigcup \Gamma_k$ is ordered.

Theorem 2. *A free distributive ΩA-group G with any system of free generators X can be ordered by extending the lexicographic linear order relation in the absolutely free algebra $\Gamma(X, \Omega)$, defined by arbitrarily prescribed order relations in the sets X and Ω.*

Corollary. *A free nonassociative ring K with any set of free generators X can be ordered by extending the lexicographic order in the absolutely free groupoid $\Gamma(X, \cdot)$, defined by an arbitrarily prescribed order relation on the set X.*

No. 3. **Free distributive unary Ω-groups.** Let Ω be an arbitrary system of unary operations. Take some nonempty set of symbols X and construct the absolutely free algebra $\Gamma(X, \Omega)$. We take the set of elements of this algebra as the system of free generators of the additive free group F. If $w \in F$, $w \neq 0$, then w has a unique representation of the form $w = k_1\gamma_1 + \dots + k_n\gamma_n$, where $\gamma_i \in \Gamma(X, \Omega)$, $\gamma_i \neq \gamma_{i+1}$, and k_i are nonzero integers. We put $w\omega = k_1(\gamma_1\omega) + \dots + k_n(\gamma_n\omega)$ and $0\omega = 0$ for every $\omega \in \Omega$. With this definition in the group F of unary operations from Ω, the group F becomes a distributive Ω-group. It is easy to see that it will be free in the class of all distributive Ω-groups with the given system of unary operations Ω. We shall call F a **free distributive unary Ω-group**.

Using Hall' s theory of basic commutators ([10]) (see also A. I. Shirshov ([11])), one can prove the following assertion.

Theorem 3. *A free distributive unary Ω-group F with any system of free generators X can be ordered by extending the lexicographic linear order relation in the absolutely free algebra $\Gamma(X, \Omega)$, defined by arbitrarily prescribed order relations in the sets X and Ω.*

No. 4. **Free distributive quasirings.** A distributive Ω-group G, whose system of multioperators Ω consists only of one binary operation—multiplication—will be called a **distributive quasiring** (cf. ([12])). The class K of distributive quasirings is characterized by a system of identities consisting of identities defining the class of additive groups and two distributivity identities:

$$(x+y)z = xz + yz, \qquad z(x+y) = zx + zy.$$

Consequently, the class K is **primitive**, and one can speak of **free distributive quasirings**.

A free distributive quasiring F with a system of free generators X can be constructed in the following way. First we construct the following free algebras, each time taking the set X as the system of free generators: 1) an absolutely free multiplicative groupoid Γ, in which we distinguish the subgroupoid $Y = \Gamma \setminus X$; 2) a free additive group F_1; 3) a free nonassociative ring G,

in which we single out the subring F_2, generated by the set Y, and the subgroup G_1, generated by the set X. For the groups F_1, G_1, F_2, G the following decompositions hold:

$$F_1 = \sum_{x \in X} * \{x\}, \qquad G_1 = \sum_{x \in X} \{x\}, \qquad F_2 = \sum_{y \in Y} \{y\}, \qquad G = G_1 + F_2.$$

Let σ_1 be the natural homomorphism of the group F_1 onto the group G_1, and let σ_2 be the identity mapping of the group F_2 onto itself. By $F = F_1 * F_2$ denote the free sum of the groups F_1, F_2. There exists a uniquely determined homomorphism σ of the group F into the group G extending the homomorphisms σ_1, σ_2. Since the product of any two elements of G belongs to the subgroup F_2, we have $(f\sigma) \cdot (g\sigma) \in F_2$ for all $f, g \in F$. Put $f \cdot g = (f\sigma) \cdot (g\sigma)$. It is easy to see that the group F, together with the multiplication operation defined in this way, is a distributive quasiring.

Theorem 4. *Every mapping φ of the set X into any distributive quasiring H can be extended, and moreover uniquely, to a homomorphism of the quasiring F constructed above into the quasiring H.*

Consequently, F is a free distributive quasiring with the system of free generators X.

Theorem 5. *The free distributive quasiring F with any system of free generators X can be ordered by extending the lexicographic order relation in the absolutely free groupoid $\Gamma(X, \cdot)$, defined by an arbitrarily prescribed order relation on the set X.*

No. 5. Free modules. Let R be the group ring of some multiplicative group G. A free R-module S, considered as a G-module, is called a **free G-module**.

Theorem 6. *A free G-module S with basis X over an orderable group G can be ordered by extending any order relation on the set X.*

Let K be some class of Ω-groups. An Ω-group H is called **K-approximable** [13] if, for every element $h \in H,\ h \neq 0$, there exists an Ω-homomorphism σ of the Ω-group H into an Ω-group from the class K, under which the image $h\sigma$ of the element h is also distinct from zero. If the class K consists of finite Ω-groups (or of finite Ω-groups whose order is equal to a power of a given prime number p), then K-approximability is called finite (or, respectively, approximability by finite p-groups).

Theorem 7. *If the group G is finitely approximable, then any free G-module S is approximable by finite p-groups for any prime number p.*

No. 6. Triangular extensions. Let G be a multiplicative group and S some free G-module. The symbols (g, s), where $g \in G,\ s \in S$, with respect to the multiplication

$$(g_1, s_1) \cdot (g_2, s_2) = (g_1 g_2,\ s_1 g_2 + s_2)$$

form a group, which we shall call the **triangular extension** of the group G and denote by GS (cf. [10], p. 256).

Theorem 8. *If the group G is orderable (or finitely approximable), then any triangular extension GS of it is also orderable (or, respectively, finitely approximable).*

No. 7. Free groups. Using Theorem 8 and one result of M. Hall (see [10], Lemma 15.5.1), it is easy to prove the following assertion:

Theorem 9. *If the quotient group F/A of a free group F by some normal divisor A is orderable (or finitely approximable), then the quotient group $F/A^{(n)}$ of the group F by any n-th commutant $A^{(n)}$ of the group A is also orderable (or, respectively, finitely approximable).*

As an immediate consequence of this theorem, let us note the following known result.

Corollary. *The free solvable group $F/F^{(n)}$ is orderable and finitely approximable.*

Finite approximability of the free solvable group was first proved by Gruenberg in paper [14]. In the same paper a theorem of P. Hall was formulated without proof, from which the orderability of the free solvable group follows.

Let us note that the factor group $F/[A, A]$ of a free group F, being a torsion-free group for any normal divisor A of the group F [15], may turn out to be non-orderable. Therefore the assumption on the orderability of the factor group F/A in the condition of Theorem 9 is essential.

Example. Let F be the free group freely generated by the elements u, v, and let B be the cyclic group of order 2 with generator b. The mapping $u \to b,\ v \to 1$ of the set of free generators of the group F into the group B extends (and moreover uniquely) to a homomorphism of the group F onto the group B. Let A be the kernel of this homomorphism. Using the Schreier method, it is easy to show that the group A is freely generated by the elements u^2, v, uvu^{-1}. Let $f = vuv^{-1}$, $g = v^{-1}uv$. Since the factor group $A/[A, A]$ is commutative, $f^2 \equiv g^2 \pmod{[A, A]}$. However $f \not\equiv g \pmod{[A, A]}$, since otherwise we would have $(uvu^{-1})^2 \equiv v^2 \pmod{[A, A]}$, which is impossible in view of the linear independence of the elements uvu^{-1}, v modulo $[A, A]$. Thus the group $F/[A, A]$ is not a group with unique extraction of roots and, consequently, cannot be ordered.

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Note: Figure translations are in progress. See original paper for figures.