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Abstract

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ON DEFECTS OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

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We shall adhere to the terminology and notation standard in Nevanlinna theory. By ρ we shall denote the order of the meromorphic function $f(z)$ under consideration, and by λ its lower order.

Theorem 1. *In order that, for given a and b, $a \neq b$, there exist a meromorphic function of lower order $\lambda < 1$ with $\delta(a) = 1 - x$, $\delta(b) = 1 - y$, it is necessary and sufficient that x and y satisfy the following relations*:
1) $1 \geq x \geq -E[(y - \cos \pi\lambda)/2]$;
2) $1 \geq y \geq -E[(x - \cos \pi\lambda)/2]$;
3) $x^2 + y^2 - 2xy \cos \pi\lambda \geq \sin^2 \pi\lambda$.

Edrei and Fuchs ([3]) proved an analogous theorem in which ρ stands in place of λ. If $1 > \rho > \lambda$, then our theorem gives more precise information than the theorem of Edrei and Fuchs, since, when λ is replaced by ρ in the system of inequalities 1)–3), the set of solutions is enlarged. The case $\rho \geq 1 > \lambda$ is not covered at all by the theorem of Edrei and Fuchs.

The method of ([3]) is essentially based on the representation of a function with $\rho < 1$ by a canonical product of genus zero (which functions with $\lambda < 1$, $\rho \leq \infty$, generally speaking, do not admit) and on a lemma on the growth of monotone functions of finite order due to Pólya. For our purposes the method of ([3]) had to be substantially modified. To this end we used: a) an important idea of Kjellberg ([5]), which consists in the fact that, for $\lambda < 1$ and for some arbitrarily large values $R > 0$, the function $f(z)$ in the disk $|z| \leq R$ can be represented in the form $f(z) = \alpha(z)\omega(z)$, where the first factor has genus zero and the second is very close to 1; b) a lemma established by us on the growth of monotone functions, replacing Pólya' s lemma.

We note that the sufficiency of the conditions of Theorem 1 was proved in ([3]), since the examples constructed there of meromorphic functions with $\delta(a) = 1 - x$, $\delta(b) = 1 - y$ have the property $\lambda = \rho$. For the case $\lambda = 0$ Edrei and Fuchs proved ([4]) also the necessity of the conditions of the theorem.

1°. In what follows we shall assume that $a = \infty$, $b = 0$, and that the meromorphic function $f(z)$ under consideration satisfies the condition $f(0) = 1$. This will not affect the generality of our arguments. Let a_k be the zeros of $f(z)$, and b_l its poles. Taking $R > 0$, put

$$\alpha(z) = \prod_{|a_k| \le R} \left(1 - \frac{z}{a_k}\right) \prod_{|b_l| \le R} \left(1 - \frac{z}{b_l}\right), \qquad \omega(z) = \frac{f(z)}{\alpha(z)},$$

$$H(r) = \sum_{|a_k| \le R} \ln \left(1 + \frac{r}{|a_k|}\right) \sum_{|b_l| \le R} \ln \left(1 + \frac{r}{|b_l|}\right)$$

(all these quantities, of course, depend on the choice of R). By the letter C, with subscripts, we shall denote positive constants which, generally speaking, depend on the function $f(z)$.

Lemma 1. *For $r \le R$ the inequality holds*

$$H(r) \le C_1 T(2R, f).$$

Lemma 2 (cf. ([5])). *For $r \le 0.5R$, $0 \le \varphi \le 2\pi$, the inequality holds*

$$\left|\ln \omega(re^{i\varphi})\right| \le C_2 r R^{-1} T(2R, f).$$

* $E[x]$ denotes the integer part of the number x.

The proof is based on the representation of $\ln \omega(z)$ in the disk $|z| \le R$ by Schwarz' s formula; Lemma 1 is used.

Lemma 3. *Let $V(r) \ge 0$ be a continuous nondecreasing function of r, $1 \le r < \infty$, and let $\sigma > 0$ be such that:*

$$\alpha)\ \lim_{r\to\infty} V(r) r^{-\sigma} = 0, \qquad \beta)\ \overline{\lim}_{r\to\infty} V(r) r^{-\sigma} = \infty.$$

Fix arbitrarily ε, $0 < \varepsilon < 1$. Then there exist two sequences $r_n \uparrow \infty$ and $R_n \uparrow \infty$ such that $(n = 1, 2, ...)$

$$\text{a) } r_{nR} n^{-1} \le \varepsilon, \qquad \text{b) } V(r_n) r_n^{-\sigma} = \max_{1 \le r \le R_n} [V(r) r^{-\sigma}].$$

Proof. By virtue of α), there exists a sequence $R_n \uparrow \infty$ such that

$$\lim_{n\to\infty} V(R_n) R_n^{-\sigma} = 0.$$

Since $V(r)$ is nondecreasing, we have

$$\lim_{n\to\infty}\left\{\max_{\varepsilon R_n\le r\le R_n}[V(r)r^{-\sigma}]\right\}=0. \tag{1}$$

Denote by F the closed set

$$\{r: V(r)r^{-\sigma}=\max_{1\le t\le r}[V(t)t^{-\sigma}]\}.$$

This set, by virtue of β), is unbounded, and on it $V(r)r^{-\sigma}\uparrow\infty$. Therefore it follows from (1) that for $n\ge n_0$ the segments $[\varepsilon R_n, R_n]$ do not intersect the set F. Denote by r_n $(n\ge n_0)$ the point of the set $F\cap[1,\varepsilon R_n]$ nearest to the point εR_n. It is easy to verify that the sequences $\{r_n\}$, $\{R_n\}$ satisfy a), b), starting with $n=n_0$.

Lemma 4 (Edrei and Fuchs ([2])). *Let $f(z)$ $(f(0)=1)$ be a meromorphic function of genus zero. The quantity $\beta=\beta(r,f)$, $0\le\beta\le\pi$, can be chosen in such a way that the relation*

$$T(r,f)\le\int_0^\infty N(t,f)K(t,r,\beta)\,dt+\int_0^\infty N(t,f^{-1})K(t,r,\pi-\beta)\,dt,$$

holds, where

$$K(t,r,\beta)=\pi^{-1}r\sin\beta\,(t^2+r^2-2rt\cos\beta)^{-1}$$

is the Poisson kernel for the upper half-plane.

2°. **Proof of Theorem 1** (necessity). If $\lambda=\rho$, the assertion follows from the result ([3]), therefore it suffices to consider the case $\lambda<\rho$. From Lemma 2 it follows that, for $r\le 0.5R$,

$$T(r,f)\le T(r,\alpha)+T(r,\omega)\le T(r,\alpha)+C_2rR^{-1}T(2R,f).$$

We apply the inequality of Lemma 4 to $T(r,\alpha)$. Taking into account in it that

$$N(t,\alpha)=N(t,f)\quad\text{for } t\le R,$$

$$N(t,\alpha)=N(R,f)+n(R,f)\ln(tR^{-1})\quad\text{for } t>R,$$

and that analogous equalities are valid for $N(t,\alpha^{-1})$, we obtain, with the help of simple estimates, the relation $(r\le 0.5R)$

$$T(r,f)\le\int_0^R N(t,f)K(t,r,\beta)\,dt+\int_0^R N(t,f^{-1})K(t,r,\pi-\beta)\,dt+C_3rR^{-1}T(2R,f). \tag{2}$$

Let now $u>1-\delta(\infty)$, $v>1-\delta(0)$; then for $t\ge c\ge 1$ we have the inequalities

$$N(t,f)\le uT(t,f),\qquad N(t,f^{-1})\le vT(t,f).$$

Using them in (2), we shall have ($r \leq 0.5R$)

$$T(r,f) \leq u \int_c^R T(t,f) K(t,r,\beta)\, dt + v \int_c^R T(t,f) K(t,r,\pi-\beta)\, dt +$$

$$+ C_3 r R^{-1} T(2R,f) + C_4 r^{-1}. \qquad (3)$$

Take an arbitrary σ satisfying the condition $\lambda < \sigma < \min(1,\rho)$. With this σ, Lemma 3 is applicable to $T(r,f)$. Putting in (3) $r = r_n$, $2R = R_n$, we, by virtue of a) and b), obtain a relation which, after division of both sides ···

parts of $T(r_n, f)$ has the form

$$1 \leq u \int_c^{R_n} (tr_n^{-1})^\sigma K(t,r_n,\beta)\, dt + v \int_c^{R_n} (tr_n^{-1})^\sigma K(t,r_n,\pi-\beta)\, dt + C_5 \varepsilon^{1-\sigma} + C_4 [r_n T(r_n,f)]^{-1}.$$

This inequality will obviously remain valid if c is replaced by 0, and R_n by ∞. But then the integrals are easily evaluated, and we obtain the inequality

$$\sin \pi\sigma \leq u \sin \beta\sigma + v \sin(\pi-\beta)\sigma + C_6 \varepsilon^{1-\sigma} + C_7 [r_n T(r_n,f)]^{-1}.$$

Taking into account that

$$u \sin\beta\sigma + v \sin(\pi-\beta)\sigma \leq \left[(u - v\cos\pi\sigma)^2 + (v \sin\pi\sigma)^2\right]^{1/2},$$

we arrive at the relation

$$\sin\pi\sigma \leq [u^2 + v^2 - 2uv\cos\pi\sigma]^{1/2} + C_6 \varepsilon^{1-\sigma} + C_7 [r_n T(r_n,f)]^{-1}.$$

If here we let $n \to \infty$, and then $\varepsilon \to 0$, $u \to 1 - \delta(\infty)$, $v \to 1 - \delta(0)$, $\sigma \to \lambda$, then we obtain that the quantities $x = 1 - \delta(\infty)$, $y = 1 - \delta(0)$ must satisfy inequality 3). Since always $0 \leq x \leq 1$, $0 \leq y \leq 1$, in the case $1/2 \leq \lambda < 1$ the theorem has already been proved. In the case $0 \leq \lambda < 1/2$ it remains only to prove that, if $\delta(\infty) > 1 - \cos\pi\lambda$, then necessarily $\delta(0) = 0$, and (equivalently) that, if $\delta(0) > 1 - \cos\pi\lambda$, then necessarily $\delta(\infty) = 0$. We shall prove a stronger assertion.

3°. **Theorem 2** (a generalization of one result of (1)). If $\lambda < 1/2$, then

$$\overline{\lim_{r\to\infty}} \ln^+ \mu(r,f)[T(r,f)]^{-1} \geq \pi\lambda(\operatorname{cosec}\pi\lambda)[\cos\pi\lambda - 1 + \delta(\infty)],$$

where

$$\mu(r, f) = \min_{|z|=r} |f(z)|.$$

Corollary (a generalization of a theorem of A. A. Gol' dberg (2)). If $\delta(\infty) > 1 - \cos\pi\lambda$, then there exists a sequence of circles $|z| = t_n \uparrow \infty$ on which $f(z)$ tends uniformly to ∞. Therefore ∞ is the unique deficient value.

Proof of Theorem 2. For $0 < \xi < \eta < R$, $0 < \sigma < 1$, we have (1)

$$\int_\xi^\eta \left\{\ln\mu(r, \alpha) + \frac{\pi\sigma}{\sin\pi\sigma}[N(r, f) - \cos\pi\sigma\, N(r, f^{-1})]\right\} r^{-1-\sigma}\, dr \geq C_8 H(\xi)\xi^{-\sigma} - C_9 H(\eta)\eta^{-\sigma}. \tag{4}$$

By Lemma 2, for $0 < \xi < \eta \leq 0.5R$,

$$\int_\xi^\eta |\ln\mu(r, \infty)|\, r^{-1-\sigma}\, dr \leq C_{10}(1-\sigma)^{-1}\eta^{1-\sigma}R^{-1}T(2R, f). \tag{5}$$

Taking into account that $\mu(r, f) \geq \mu(r, \alpha)\mu(r, \infty)$ and estimating H with the aid of Lemma 1, from (4) and (5), with $\eta = 0.5R$, we obtain

$$\int_\xi^\eta \left\{\ln^+\mu(r, f) + \frac{\pi\sigma}{\sin\pi\sigma}[N(r, f) - \cos\pi\sigma\, N(r, f^{-1})]\right\} r^{-1-\sigma}\, dr \geq C_8 H(\xi)\xi^{-\sigma} - C_{11}(1-\sigma)^{-1}R^{-\sigma}T(2R, f). \tag{6}$$

Now choose σ so that $\lambda < \sigma < 1$. However large ξ may be, we can choose the quantity $R = 2\eta$ in such a way that the right-hand side of (6) is positive. It follows that

$$\overline{\lim_{r\to\infty}}\{\ln^+\mu(r, f) + \pi\sigma(\operatorname{cosec}\pi\sigma)[N(r, f) - \cos\pi\sigma N(r, f^{-1})]\} \geq 0.$$

This relation is valid for any meromorphic function of lower order λ. Let us apply it to the function $f(z) - a$, where the complex number $a \neq \infty$ is chosen so that

$$\lim_{r\to\infty} N(r, (f-a)^{-1})[T(r, f)]^{-1} = 1$$

(the existence of such an a is ensured by a known theorem of Valiron). Since $\mu(r, f-a) = \mu(r, f) + O(1)$, $N(r, f-a) = N(r, f)$, $N(r, (f-a)^{-1}) = T(r, f)(1 + o(1))$, it follows, on dividing both sides by $T(r, f)$, that we obtain the relation

$$\lim_{r\to\infty} \ln^+\mu(r, f)\,[T(r, f)]^{-1} \geq \pi\sigma(\operatorname{cosec}\pi\sigma)[\cos\pi\sigma - 1 + \delta(\infty)].$$

It remains to let σ tend to λ.

Remark. By the same method Theorem 2 was proved simultaneously and independently of the author by A. A. Goldberg. With the aid of this method one can show that Theorems 2.3, 2.4, 2.5, 2.6, and 2.7 of work ([1]) remain valid* if the quantity $\tilde{\lambda}$ occurring in them is replaced by λ and the requirement of the presence of zeros is omitted. This circumstance was also established simultaneously and independently by A. A. Goldberg.

4°. Put

$$\varkappa(f) = \lim_{r\to\infty} [N(r, f) + N(r, f^{-1})] \, [T(r, f^{-1})]^{-1}.$$

Nevanlinna showed ([6]) that $\varkappa(f) \geq d(\rho)$, where $d(\rho)$ is a quantity positive for nonintegral ρ, and posed the problem of finding the best lower estimate of $\varkappa(f)$ in terms of ρ. For $0 \leq \rho \leq 1$ the best estimate turned out to be ([3]) $\varkappa(f) \geq 1$ for $0 \leq \rho \leq \frac{1}{2}$, $\varkappa(f) \geq \sin \pi\rho$ for $\frac{1}{2} \leq \rho \leq 1$. For $1 < \rho < \infty$ an estimate ([4]), differing little from the exact one, was obtained:

$$\varkappa(f) \geq |\sin \pi\rho| \, \{2, 2\rho + 0, 5|\sin \pi\rho|\}.$$

With the aid of the method used in the proof of Theorem 1, one can obtain a lower estimate of $\varkappa(f)$ in terms of λ and ρ, valid for $\lambda < \infty$, $\rho \leq \infty$, and containing estimate ([3]).

Theorem 3. Define the function $\nu(x)$, $0 \leq x < \infty$, as follows: $\nu(x) = 1$ for $0 \leq x \leq \frac{1}{2}$; $\nu(x) = \sin \pi x$ for $\frac{1}{2} \leq x \leq 1$; $\nu(x) = |\sin \pi x| \, \{6x + 0, 5|\sin \pi x|\}^{-1}$ for $1 \leq x < \infty$. The estimate

$$\varkappa(f) \geq \max_{\lambda \leq x \leq \rho} \nu(x)$$

is valid.

In the proof, the function $\alpha(z)$ in the decomposition $f(z) = \alpha(z)\omega(z)$ is constructed with the aid of the canonical Weierstrass products $E(z, q)$, $q = E[x]$, $\lambda < x < \rho$; the estimate for $m(r, E)$, $m(r, E^{-1})$ from ([4]) and Lemma 3 are used. Theorem 3 contains the result of Edrei and Fuchs ([4]): if $\varkappa(f) = 0$, then $\lambda = \rho =$ a natural number.

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[*] From the formulation of Theorem 2.5 one should exclude the erroneous relation (2.20′).